lume XII July 1958 Number 63

UNIVERSITY
OF MICHIGAN
APR 29 1959
MATHEMATICS
BRARY

Mathematical Tables
and other
Aids to Computation

Published Quarterly
by the
National Academy of Sciences—National Research Council

Published quarterly in January, April, July, and October by the National Academy of Sciences—National Research Council, Prince and Lemon Sts., Lancaster, Pa., and Washington, D. C.

Second-class postage paid at Lancaster, Pa.

Editorial Committee

Division of Mathematics

National Academy of Sciences—National Research Council

Washington, D. C.

C. B. TOMPKINS, Chairman, University of California, Los Angeles, California

C. C. CRAIG, University of Michigan, Ann Arbor, Michigan

ALAN FLETCHER, University of Liverpool, Liverpool 3, England

EUGENE ISAACSON, New York University, New York 3, New York

A. H. TAUB, University of Illinois, Urbana, Illinois

C. V. L. SMITH, Ballistic Research Labs., Aberdeen Proving Ground, Maryland

• • •

Information to Subscribers

The journal is published quarterly in one volume per year with issues numbered serially since Volume I, Number 1. Subscriptions are $5.00 per year, single copies $1.50. Back issues are available as follows:

Volume I (1943–1945), Nos. 10 and 12 *only* are available; $1.00 per issue.

Volume II (1946–1947), Nos. 13, 14, 17, 18, 19, and 20 *only* available; $1.00 per issue

Volume III (1948–1949), Nos. 21–28 available. $4.00 per year (four issues), $1.25 per issue

Volume IV (1950 and following), all issues available; $5.00 per year, $1.50 per issue

All payments are to be made to the National Academy of Sciences and forwarded to the Publications Office, 2101 Constitution Avenue, Washington, D. C.

Agents for Great Britain and Ireland: Scientific Computing Service, Ltd., 23 Bedford Square, London W.C.1

• • •

Microcard Edition

Volumes I–X (1943–1956), Nos. 1–56 are now available on Microcards and may be purchased from The Microcard Foundation, Box 2145, Madison 5, Wisconsin, at a cost of $20.00 for the complete set. Future volumes will be available on Microcards in the year following original publication.

• • •

Information to Contributors

All contributions intended for publication in *Mathematical Tables and Other Aids to Computation* and all books for review should be addressed to C. B. Tompkins, Department of Mathematics, University of California, Los Angeles 24, California. The author should mention the name of an appropriate editor for his paper if this is convenient.

An Inverse Method for The Generation of Random Normal Deviates on Large-Scale Computers

By Mervin E. Muller

1. Introduction. Many applications of electronic computers require efficient generation of large numbers of random normal deviates; see, e.g. [7], [12], [14], [17]. Tables of random normal deviates are of course available, for example [18], [23], but they are not sufficiently extensive for many purposes and an outside source of this kind cannot usually be used effectively by the computer. What is required is some method of generation which can be rapidly carried out by the machine itself.

Methods are available by which pseudo random numbers may be produced, see for example [2], [5], [6], [10], [11], [13], [20], [22]. Judging by the results given, for example, by [5], [6], [20], [21], the most satisfactory procedure now in use for generating random numbers is that based on "residue class" techniques, see for example [13], [20]. We shall not consider here the validity of these methods but shall assume that some satisfactory method of producing uniform random deviates is available from which random normal deviates are to be produced. Several ways of generating random normal deviates are known; see, for example, [1], [6], [21], but as is shown in [15], these methods are not as fast nor as reliable, especially for extreme tail values, as that developed here.

In this paper the inverse of the Normal Distribution Function is approximated stepwise by rational functions. Hence a uniform random deviate may be transformed into a normal deviate. The details of the approximations have been obtained for use on a large scale binary machine with index registers, where it is possible to utilize memory space in order to obtain greater speed.

The present approach can be used, if desired, in the course of generating arbitrary n-variate normal deviates or Chi-squared variates.

2. Inverse method. In principle, the inverse method of generating a normal deviate x from a uniform deviate u is well known. The problem is to find the inverse relationship $x = x(u)$ given that

$$u = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.$$

The actual determination of $x(u)$ offers certain numerical difficulties when it is desired to generate reliable normal deviates, especially for large values of x. We have considered this problem for two aspects. First, speed and secondly and equally important, reliability. We have insisted that the maximum absolute error in x should be less than 4×10^{-4} in the range $-5 \leq x \leq 5$, where we assume

Received October 26, 1957. Prepared in connection with research sponsored by the Office of Ordnance Research, U. S. Army; Statistical Techniques Research Group, Princeton University, Contract No. DA 36-034-ORD 2297.

u is correct, and where Prob $\{-5 \leq X \leq 5\} > 1 - 6 \times 10^{-7}$ so that x is correct to within 4×10^{-4} except for an event of probability less than 6×10^{-7}.

The relation $x = x(u)$ is approached in stages. The interval $[0, 1]$ for u is subdivided so that over each sub-interval it is possible to obtain a reliable and fast procedure for computing x. Over most of $(0, 1)$ $x = x(u)$ is approximated by Chebyshev polynomials. As x becomes large in absolute value it is necessary to increase the degree of the approximating polynomial. However, even though the degree of the polynomial increases, the frequency with which these approximations are needed decreases; hence this method will use, on the average, a low order approximating polynomial for $x = x(u)$. Due to symmetry, it is actually only necessary to study $x = x(u)$ for $1/2 \leq u \leq 1$.

For speed in computing it is best to have the Chebyshev polynomials of as low degree as possible, subject to the specified level of accuracy. Following this approach one could find sub-intervals of greatest or optimum width, see [3]. Two drawbacks then become apparent. If on a binary computer the sub-interval widths are not some negative power of two, considerable computing time is spent in mapping a given sub-interval onto the range $(-1, 1)$ which is needed when using a Chebyshev approximation. More important, if the intervals are not of uniform width and a negative power of two, considerable machine time, and memory space, is required to select the appropriate approximation. For, it will not then be possible to use an index register to select the appropriate approximation. Taking these facts into consideration, and taking into account memory space requirements, it was decided to have the widths of the sub-intervals equal $1/128$. With this choice of interval width the largest part of $(0, 1)$, namely $1/8 \leq u \leq 7/8$, could be approximated by linear functions. Quadratic and quartic approximations are used for $7/8 < u \leq 127/128$. For $127/128 < u < 1$, and by symmetry $0 < u < 1/128$, $x = x(u)$ has a singularity of logarithmic type; consequently, for u in this sub-interval an approximation of more subtle type than Chebyshev polynomials is needed.

3. Use of polynomials. Since the techniques used in approximating $x = x(u)$ have possible application when considering the generation of other possible pseudo random variables we shall review the techniques in detail.

To approximate $x = x(u)$ in the jth sub-interval,

$$\left(\frac{64 + j - 1}{128} \leq u \leq \frac{64 + j}{128}\right), \quad j = 1(1)63,$$

by Chebyshev polynomials we shall make use of the technique of Interpolating Chebyshev Polynomials as developed by C. Lanczos, see for example [8], [9]. To begin, the jth sub-interval is transformed onto $(-1, 1)$ by using

$$r = 256u - 127 - 2j.$$

Let $r = \cos\theta$, $0 \leq \theta \leq \pi$. Let $T_i(r) = \cos[i(\cos^{-1} r)]$ denote the ith Chebyshev polynomial. Thus for the jth sub-interval, $x = x_j(r)$ will be approximate by a polynomial of nth degree, namely,

$$x_{nj}(r) = 1/2\, d_{0j} + \sum_{i=1}^{n} d_{ij} T_i(r);$$

where the d_{ij}'s are obtained from

$$d_{ij} = \frac{2}{n+1} \sum_{k=0}^{n} x(r_k) T_i(r_k),$$

and where the r_k's are the zeros of the $(n+1)$th Chebyshev polynomial. Thus $r_k = \cos(\pi(2k+1)/2(n+1))$. This approach approximately minimizes the maximum error throughout the sub-interval.

For computing purpose it is more convenient to express the $T_i(r)$'s as polynomials in r, see Lanczos [8], [9], for example $T_1(r) = r$, $T_2(r) = 2r^2 - 1$. One then obtains

$$x_{nj}(r) = \sum_{i=0}^{n} a_{ij} r^i. \tag{1}$$

The coefficients a_{ij} are given in section 4.

In calculating the d_{ij}'s, and hence the a_{ij}'s, it is necessary to evaluate $x = x(u)$ for u_{kj}'s where

$$u_{kj} = \frac{r_k + 127 + 2j}{256},$$

and this was done for the seven intervals $j = 57(1)63$, i.e., $7/8 < u \leq 127/128$. The values of $x(u_{kj})$ were obtained by cubic inverse interpolation in the National Bureau of Standards Tables of the Normal Probability Function, [19].

However, for the intervals, $j = 1(1)56$, i.e., $1/2 \leq u \leq 7/8$, where the linear approximations are adequate the essential computations for the a_{ij}'s are concerned with finding the inverse values, namely $x = x(u_{kj})$. However, for the linear cases the a_{ij}'s given in section 4 were not obtained in the above manner since we had previously needed to have values of x corresponding to $u_s = (128 + s)/256$, $s = 0, 1, 2, \cdots, 128$. Consequently we avoided the additional labor of finding the necessary x's by utilizing the available $x(u_s)$'s. These values are included as Table 5 in the Appendix. Thus, while not fitting the straight lines by first degree Chebyshev polynomials we were able to obtain the desired level of precision as follows: In the jth interval, where it is appropriate to fit a straight line, consider u_j, u_{j+1}, and $u_{j,1/2} = (u_j + u_{j+1})/2$. Let x_j, x_{j+1}, $x_{j,1/2}$ denote the corresponding x values. The jth line is fitted such that the deviations at the ends of the class interval are equal and such that the deviation at the mid-point, namely $u_{j,1/2}$, is equal in magnitude to the deviations at the end points, but is of opposite sign. This approach gives essentially a Chebyshev-type approximation in that we are striving to minimize the maximum error in a given sub-interval. By straightforward computations one then obtains that

$$a_{0j} = \frac{2(x_{j+1} + x_{j,1/2}) - (x_{j+1} - x_j)(255 + 4j)}{4},$$

$$a_{1j} = 128(x_{j+1} - x_j).$$

In the selection of the width of a sub-interval, w_j, or for determining the appropriate degree of the Chebyshev Polynomial for a given sub-interval, the following

result is used, see [3] for more general details,

$$w_j = 4\left[\frac{(n+1)!}{2}\epsilon|x^{(n+1)}(c_j)|^{-1}\right]^{\frac{1}{n+1}},$$

where ϵ is the largest absolute error that is tolerable, in our case $\epsilon = 4 \times 10^{-4}$, and where $x^{(n+1)}(u)$ is the $(n+1)$st derivative of $x = x(u)$ with respect to u.

4. Table of coefficients. The following tables provide the necessary coefficients for the approximations given by equation (1) of section 3.

TABLE 1. *Linear Cases*

j	a_{0j}	a_{1j}	j	a_{0j}	a_{1j}
1	−1.25339 449	2.50678 851	29	−1.56607 868	2.98461 325
2	−1.25388 344	2.50775 044	30	−1.59698 101	3.02714 356
3	−1.25487 723	2.50967 688	31	−1.63039 945	3.07264 746
4	−1.25639 365	2.51257 300	32	−1.66656 010	3.12136 699
5	−1.25845 203	2.51644 669	33	−1.70571 765	3.17357 468
6	−1.26107 329	2.52130 843	34	−1.74815 989	3.22957 838
7	−1.26428 016	2.52717 152	35	−1.79421 327	3.28972 698
8	−1.26809 727	2.53405 216	36	−1.84424 956	3.35441 736
9	−1.27255 126	2.54196 946	37	−1.89869 401	3.42410 298
10	−1.27767 105	2.55094 571	38	−1.95803 516	3.49930 401
11	−1.28348 794	2.56100 644	39	−2.02283 720	3.58062 028
12	−1.29003 594	2.57218 075	40	−2.09375 508	3.66874 687
13	−1.29735 187	2.58450 135	41	−2.17155 347	3.76449 389
14	−1.30547 572	2.59800 495	42	−2.25713 085	3.86881 137
15	−1.31445 096	2.61273 256	43	−2.35154 997	3.98282 083
16	−1.32432 485	2.62872 974	44	−2.45607 734	4.10785 604
17	−1.33514 882	2.64604 704	45	−2.57223 456	4.24551 603
18	−1.34698 984	2.66475 763	46	−2.70077 415	4.39645 556
19	−1.35986 540	2.68485 495	47	−2.84833 659	4.56815 521
20	−1.37390 816	2.70651 014	48	−3.01223 276	4.75713 372
21	−1.38914 726	2.72973 042	49	−3.19590 941	4.96703 562
22	−1.40567 396	2.75461 642	50	−3.40829 673	5.20760 458
23	−1.42357 635	2.78126 053	51	−3.65403 652	5.48350 516
24	−1.44295 134	2.80976 489	52	−3.93953 404	5.80124 828
25	−1.46390 578	2.84024 264	53	−4.27256 248	6.16869 364
26	−1.48655 769	2.87281 914	54	−4.67044 630	6.60394 516
27	−1.51103 773	2.90763 359	55	−5.15058 375	7.12472 205
28	−1.53749 093	2.94484 078	56	−5.74560 472	7.76467 397

TABLE 2. *Quadratic Cases*

j	a_{0j}	a_{1j}	a_{2j}
57	1.56668 859	−0.03343 48405	.00087 5575
58	1.63732 538	−0.03745 15701	.00114 804
59	1.71722 812	−0.04284 26652	.00157 546
60	1.80989 233	−0.05049 00254	.00230 549
61	1.92135 077	−0.06226 30013	.00372 027
62	2.06352 790	−0.08299 31005	.00708 977

TABLE 3. *Quartic Case*

j	a_{0j}	a_{1j}	a_{2j}	a_{3j}	a_{4j}
63	+2.26622 681	+0.12757 931	+0.01844 432	+0.00424 42872	+0.00104 06032

5. The interval $(127/128 < u < 1)$. In view of the fact that $x(u)$ may be looked upon as a function having a singularity of logarithmic type in this interval, it is necessary to abandon the use of a polynomial type approximation here. A satisfactory rational approximation is obtained by using a truncated continued fraction expansion, see for example [4], [16]. For any given value of u the corresponding value of x is approximated by the following recurrence relation. The kth convergent to $x(u)$ is given by

$$x_k(u) = \frac{M_k(u)}{N_k(u)},$$

where $M_k(u)$ and $N_k(u)$ are determined as follows:

$$(2)\quad M_{k+1}(u) = d_k M_k(u) + (u - u_{k-1})M_{k-1}(u); \quad M_0(u) = 1; \quad M_1(u) = d_0;$$

$$(3)\quad N_{k+1}(u) = d_k N_k(u) + (u - u_{k-1})N_{k-1}(u); \quad N_0(u) = 0; \quad N_1(u) = 1;$$

and where d_k denotes a selected value of the kth inverted divided difference of $x(u)$.

In order to insure sufficient precision it is suggested that the machine program test to see if $u > .99999$. When u is greater than .99999 it is suggested that the additional significant figures for u_k, $k = 10(1)14$, be utilized. Double precision operations are not necessary if these u_k are stored with the first three nines suppressed, u must then be shifted the appropriate number of places before performing $u - u_k$.

Following [4], page 406, $x_k(u)$ can be computed from

$$(4)\quad x_k(u) = d_0 + \sum_{n=1}^{k} \frac{(-1)^{n+1}(u - u_0)(u - u_1)\cdots(u - u_{n-1})}{N_n(u)N_{n+1}(u)}.$$

To insure that $x(u)$ can be approximated to within 4×10^{-4} it is necessary to stop at an appropriate value of $k = k(u)$ since the approximation being employed is essentially a divergent expansion. Numerical evaluations verified that it is sufficient to have the machine program terminate as soon as one of the following three conditions is satisfied, namely: (i) select $x_k(u)$ as the approximation to $x(u)$

TABLE 4. *Constants for Rational Approximation*

k	u_k	d_k
0	.99223 97464	2.42000 0000
1	.99461 38540	0.01826 2366
2	.99653 30262	−0.65518 1080
3	.99813 41867	0.02399 7757
4	.99903 23968	−0.26737 1460
5	.99931 28620	0.01654 1263
6	.99966 30707	−0.14194 9840
7	.99984 08914	0.00997 32778
8	.99992 76519	−0.06004 9158
9	.99996 83287	0.00511 81541
10	.99998 66542 51	−0.02329 9296
11	.99999 45874 56	0.00241 07770
12	.99999 78875 45	−0.00863 34192
13	.99999 92066 72	0.00100 76316
14	.99999 97133 484	−0.00308 28145

if $r_k = |x_k(u) - x_{k-1}(u)| < 4 \times 10^{-4}$, (ii) select $x_k(u)$ as the approximation to $x(u)$ if $r_k - r_{k+1} < 0$, and (iii) if $k = 14$ terminate the process and select $x_{14}(u)$ as the approximation to $x(u)$.

The value of k increases as u increases in the interval $(127/128 < u < 1 - 3 \times 10^{-7})$, and consequently so does the computing time. However, this subinterval will, on the average, increase the necessary computing time very little. It should be kept in mind that while this method is very appropriate and convenient for a machine which performs "floating point" multiplication, this approach would create serious scaling difficulties for a "fixed point" mode of operation.

6. Concluding comments. A detailed comparison of this method with other known methods is given in [15]. The technique proposed in this paper yields accuracy comparable with, or better than, most previous proposals using about one-quarter the computing time while requiring about twice as many memory locations.

The author wishes to express his appreciation to Dr. G. E. P. Box, Dr. Collin Mallows, and Dr. John W. Tukey for their interest and generous comments concerning this work. The author is also indebted to Mrs. A. Schay for her excellent help with the numerical computations.

Princeton University
Princeton, New Jersey

1. G. E. P. Box & M. E. Muller, "A note on the generation of normal deviates," *Ann. Math. Stat.*, to be published.
2. George E. Forsythe, "Generation and testing of random digits at the National Bureau of Standards, Los Angeles," *Monte Carlo Method*, NBS, Applied Mathematics Series 12, U. S. Government Printing Office, Washington, D. C., 1951, p. 34–35. [*MTAC*, Rev. **42**, v. XI, 1957, p. 43–44.]
3. Joseph O. Harrison, Jr., "Piecewise polynomial approximations for large-scale digital calculations." [*MTAC*, v. III, 1949, p. 400–407.]
4. F. B. Hildebrand, *Introduction to Numerical Analysis*, McGraw-Hill Book Co., Inc., New York, 1956.
5. D. L. Johnson, "Generating and testing pseudo random numbers on the IBM type 701," *MTAC*, v. X, 1956, p. 8–13.
6. M. L. Juncosa, "Random number generation on the BRL High-Speed computing machines," Report 855, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, 1953.
7. Herman Kahn, "Applications of Monte Carlo," RAND Project Report, 1954, Revised 1956, RAND Project Report RM-1237-AEC.
8. C. Lanczos, "Trigonometric interpolation of empirical and analytical functions," *Jn. Math. Phys.*, v. 17, 1938, p. 123–199.
9. C. Lanczos, *Tables of Chebyshev Polynomials* $S_n(x)$ and $C_n(x)$, NBS, Applied Mathematics Series 9, U. S. Government Printing Office, Washington, D. C., 1952. [*MTAC*, Rev. **1103**, v. 7, 1953, p. 174.]
10. D. H. Lehmer, "Mathematical methods in large scale computing units," *Proceedings Second Symposium on Large-Scale Digital Calculating Machinery*, 1949, p. 141–146. Harvard University Press, Cambridge, Mass., 1951.
11. D. H. Lehmer, Description of "Random number generation of the BRL high-speed computing machines," *Mathematical Reviews*, v. 15, 1954, p. 559.
12. NBS, Applied Mathematics Series 12, Monte Carlo Methods, 1951, p. 34–35.
13. J. Moshman, "The generation of pseudo-random numbers on a decimal calculator," *Jn.*, Assoc. for Computing Machinery, v. 1, 1954, p. 88–91.
14. M. E. Muller, "Some continuous Monte Carlo methods for the Dirichlet problem," *Ann. Math. Stat.*, v. 27, p. 569–589.
15. M. E. Muller, "A comparison of methods for generating normal deviates," Technical Report No. 9, Statistical Techniques Research Group, Department of Mathematics, Princeton University, to be published.
16. N. E. Norlund, *Vorlesungen Uber Differenzenrechnung*, Springer, Berlin, 1924.
17. Herbert A. Meyer, Editor, *Symposium on Monte Carlo Methods*, held at the University of Florida, 1954, John Wiley and Sons, Inc., New York, 1956. [*MTAC*, Rev. **43**, v. XI, 1957, p. 44–46.]
18. The Rand Corporation, *One Million Random Digits and 100,000 Normal Deviates*, The Free Press, Glencoe, Illinois, 1955. [*MTAC*, Rev. **11**, v. X, 1956, p. 39–43.]

19. NBS, Applied Mathematics Series 23, *Tables of Normal Probability Functions*, U. S. Government Printing Office, Washington, D. C., 1953.
20. O. TAUSSKY & J. TODD, "Generation and testing of pseudo-random numbers," *Symposium on Monte Carlo Methods*, University of Florida, 1954, John Wiley and Sons, Inc., 1956, p. 15–28.
21. D. TEICHROEW, *Distribution Sampling with High-Speed Computers*, Ph.D. Thesis, University of North Carolina, 1953.
22. D. F. VOTAW, JR. & J. A. RAFFERTY, "High speed sampling," *MTAC*, v. 5, 1951, p. 1–8.
23. H. WOLD, "Random normal deviates," *Tracts for Computers*, No. XXV, Cambridge University Press, New York, 1948.

Appendix

This table gives values of $x(u)$ corresponding to the Normal Distribution for $F(x_j) = 1/2 + j/256$, $j = 1(1)127$. The values were obtained from the National Bureau of Standard Tables of the Normal Probability Function, [19], by using the inversion formula:

$$x(u) = u_0 + \frac{a}{2Q_0} + \frac{u_0}{2}\left(\frac{a}{2Q_0}\right)^2 + \frac{(2u_0^2+1)}{6}\left(\frac{a}{2Q_0}\right)^3,$$

where u_0 is the nearest tabulated entry to u, and $u - u_0 = a$, and where

$$Q_0 = \frac{1}{\sqrt{2\pi}} e^{-[x(u_0)]^2/2}.$$

TABLE 5. *Inverse Values for the Normal Distribution*

j	$F(x_j)=1/2+j/256$	x_j	j	$F(x_j)=1/2+j/256$	x_j
1	0.50390 625	0.00979 167	33	0.62890 625	0.32895 791
2	0.50781 250	0.01958 429	34	0.63281 250	0.33931 161
3	0.51171 875	0.02937 878	35	0.63671 875	0.34970 180
4	0.51562 500	0.03917 609	36	0.64062 500	0.36013 003
5	0.51953 125	0.04897 716	37	0.64453 125	0.37059 729
6	0.52343 750	0.05878 294	38	0.64843 750	0.38110 545
7	0.52734 375	0.06859 437	39	0.65234 375	0.39165 587
8	0.53125 000	0.07841 241	40	0.65625 000	0.40225 007
9	0.53515 625	0.08823 802	41	0.66015 625	0.41288 960
10	0.53906 250	0.09807 215	42	0.66406 250	0.42357 608
11	0.54296 875	0.10791 578	43	0.66796 875	0.43431 116
12	0.54687 500	0.11776 987	44	0.67187 500	0.44509 652
13	0.55078 125	0.12763 542	45	0.67578 125	0.45593 392
14	0.55468 750	0.13751 340	46	0.67968 750	0.46682 512
15	0.55859 375	0.14740 482	47	0.68359 375	0.47777 199
16	0.56250 000	0.15731 068	48	0.68750 000	0.48877 641
17	0.56640 625	0.16723 201	49	0.69140 625	0.49984 034
18	0.57031 250	0.17716 982	50	0.69531 250	0.51096 581
19	0.57421 875	0.18712 516	51	0.69921 875	0.52215 488
20	0.57812 500	0.19709 908	52	0.70312 500	0.53340 971
21	0.58203 125	0.20709 265	53	0.70703 125	0.54473 251
22	0.58593 750	0.21710 695	54	0.71093 750	0.55612 559
23	0.58984 375	0.22714 306	55	0.71484 375	0.56759 132
24	0.59375 000	0.23720 211	56	0.71875 000	0.57913 216
25	0.59765 625	0.24728 522	57	0.72265 625	0.59075 066
26	0.60156 250	0.25739 353	58	0.72656 250	0.60244 945
27	0.60546 875	0.26752 821	59	0.73046 875	0.61423 129
28	0.60937 500	0.27769 044	60	0.73437 500	0.62609 901
29	0.61328 125	0.28788 143	61	0.73828 125	0.63805 558
30	0.61718 750	0.29810 241	62	0.74218 750	0.65010 407
31	0.62109 375	0.30835 463	63	0.74609 375	0.66224 768
32	0.62500 000	0.31863 936	64	0.75000 000	0.67448 975

TABLE 5—*Continued*

j	$F(x_j)=1/2+j/256$	x_j	j	$F(x_j)=1/2+j/256$	x_j
65	0.75390 625	0.68683 375	97	0.87890 625	1.16953 661
66	0.75781 250	0.69928 330	98	0.88281 250	1.18916 435
67	0.76171 875	0.71184 220	99	0.88671 875	1.20926 123
68	0.76562 500	0.72451 438	100	0.89062 500	1.22984 876
69	0.76953 125	0.73730 400	101	0.89453 125	1.25099 172
70	0.77343 750	0.75021 538	102	0.89843 750	1.27268 865
71	0.77734 375	0.76325 304	103	0.90234 375	1.29502 241
72	0.78125 000	0.77642 176	104	0.90625 000	1.31801 090
73	0.78515 625	0.78972 652	105	0.91015 625	1.34171 784
74	0.78906 250	0.80317 257	106	0.91406 250	1.36620 382
75	0.79296 875	0.81676 542	107	0.91796 875	1.39153 749
76	0.79687 500	0.83051 088	108	0.92187 500	1.41779 714
77	0.80078 125	0.84441 508	109	0.92578 125	1.44507 258
78	0.80468 750	0.85848 447	110	0.92968 750	1.47345 903
79	0.80859 375	0.87272 589	111	0.93359 375	1.50310 294
80	0.81250 000	0.88714 656	112	0.93750 000	1.53412 054
81	0.81640 625	0.90175 411	113	0.94140 625	1.56668 859
82	0.82031 250	0.91655 667	114	0.94531 250	1.60100 866
83	0.82421 875	0.93156 283	115	0.94921 875	1.63732 538
84	0.82812 500	0.94678 176	116	0.95312 500	1.67594 192
85	0.83203 125	0.96222 320	117	0.95703 125	1.71722 812
86	0.83593 750	0.97789 754	118	0.96093 750	1.76167 041
87	0.83984 375	0.99381 591	119	0.96484 375	1.80989 233
88	0.84375 000	1.00999 017	120	0.96875 000	1.86273 187
89	0.84765 625	1.02643 306	121	0.97265 625	1.92135 077
90	0.85156 250	1.04315 826	122	0.97656 250	1.98742 789
91	0.85546 875	1.06018 048	123	0.98046 875	2.06352 790
92	0.85937 500	1.07750 557	124	0.98437 500	2.15387 469
93	0.86328 125	1.09518 065	125	0.98828 125	2.26622 681
94	0.86718 750	1.11319 428	126	0.99218 750	2.41755 902
95	0.87109 375	1.13157 656	127	0.99609 375	2.66006 747
96	0.87500 000	1.15035 938			

Algebraic Approximations for Laplace's Equation in the Neighborhood of Interfaces

By J. W. Sheldon

1. Introduction. Let it be required to solve the following problem:

Problem A:

Let C_i, $i = 1, 2$, be two simple, closed plane curves with continuous curvature. Let C_2 enclose all the points of C_1. Let G_1 be the region interior to C_1. Let G_2 be the region interior to C_2 and exterior to C_1. Let W be a continuous, bounded function of position on C_2. Let it be required to find harmonic functions $V^{(i)}$ such that

(a) $V^{(2)} = W$ on C_2.

(b) $V^{(i)}$ is regular and bounded in G_i.

Received September 19, 1957.

(c) $V^{(1)} = V^{(2)}$ on C_1.

(d) $D_1 \dfrac{\partial V^{(1)}}{\partial n} = D_2 \dfrac{\partial V^{(2)}}{\partial n}$ on C_1, where $D_i > 0$ and $\dfrac{\partial}{\partial n}$ denotes differentiation along the normal to C_1, the positive normal direction being chosen as the direction from G_1 into G_2.

Problems with boundary conditions (c) and (d) connecting distinct harmonic functions on a common boundary between adjacent regions arise in electrostatics, and specific cases are treated in books dealing with electrostatic theory. (See, for example, Jeans [1] and Smythe [2].) For electrostatics, D_1 and D_2 are the dielectric constants in G_1 and G_2. Boundary condition (c) in electrostatic theory is derived from the requirement that no work be done taking a charge around a closed path which crosses C_1. The required condition is found to be $V^{(1)} = V^{(2)}$ +"constant" and we set "constant" to zero. Boundary condition (d) is derived from Gauss's electric flux theorem. Boundary condition (d) may also be derived by a variational method. If we require that D_1 times the Dirichlet integral for G_1 plus D_2 times the Dirichlet integral for G_2 be an extremal, and that $V^{(1)} = V^{(2)}$ on C_1, then (d) is obtained as the natural boundary condition on C_1.

Problem A may also arise for steady-state heat conduction, steady-state diffusion, steady-state current flow, and for problems of incompressible, immiscible fluid flow in porous media. In the latter problem C_1 may be a moving curve [3]. Boundary conditions (c) and (d) may also arise in transient problems of diffusion type.

We suppose that we wish to solve Problem A by finite difference methods. We introduce a rectangular network over a region covering G_1 and G_2. We introduce a function φ, defined at each meshpoint interior to C_2, which is to approximate in some way the solution of Problem A. In order to determine φ we require a system of equations equal in number to the number of meshpoints interior to C_2. We call each equation in the system an "algebraic approximation." In the way that we will treat this problem, there will be an algebraic approximation associated with each meshpoint inside C_2 in the sense that each algebraic approximation approximates to the conditions of Problem A best in the neighborhood of a unique meshpoint and no two algebraic approximations approximate Problem A best in the neighborhood of the same meshpoint. The meshpoints interior to C_2 may be classified in the following way:

(i) Meshpoints adjacent to C_2. A meshpoint is adjacent to C_2 if a straight-line segment from the meshpoint to one of the four nearest neighbor meshpoints *intersects* or *extends* to C_2.

(ii) Meshpoints adjacent to C_1. A meshpoint is adjacent to C_1 if the meshpoint is situated on C_1 or if a straight-line segment from the meshpoint to one of the four nearest neighbor meshpoints *intersects* C_1 (but *not* if the segment only *extends to* C_1).

(iii) Normal meshpoints. All other meshpoints interior to C_2 are called normal meshpoints.

The algebraic approximation to be associated with the normal meshpoints is the well-known five-point formula for approximating the Laplace equation.

Algebraic approximations at the meshpoints adjacent to C_2 have been obtained. See, for example, Shortley and Weller [4] or Viswanathan [5] (Algebraic approximations have been obtained for Dirichlet or Neuman or mixed conditions on C_2.) The present paper is concerned with obtaining algebraic approximations for meshpoints adjacent to C_1.

Problems with boundary conditions (c) and (d) of Problem A have been treated by finite difference methods in connection with the "group-diffusion" problems of reactor design [6, 7]. To the author's knowledge, in this field it has always been assumed that the interfaces between materials have contours such that analogues of equations (21) or (22) apply. If the interface does not meet the conditions for equations (21) or (22), it has been customary to perturb the interface so that it does. In some problem this may be an unsatisfactory procedure. In the case of "moving boundary" problems, for example, the essence of the problem consists in getting the velocity of the moving boundary accurately.

2. Existence of solution to Problem A. Books on potential theory which give existence proofs for the Neuman and Dirichlet problems do not (to the author's knowledge) treat Problem A. An existence proof can be constructed following the integral equation technique described by Kellogg [8]. This method consists in postulating charge and dipole distributions on the boundary curves, and then obtaining and proving the existence of a solution of the equations which must be satisfied if the charge and dipole distributions are to produce potential functions which satisfy the given boundary conditions. The equations obtained are linear integral equations. The method can be extended to show that Problem A possesses a solution. It can be shown that $V^{(2)}$ is the potential of a continuous dipole distribution on C_2 plus the potential of a continuous charge distribution on C_1, $V^{(1)}$ is the potential of a continuous charge distribution on C_1. (The charge distribution on C_1 to produce $V^{(1)}$ is different from that on C_1 which contributes to produce $V^{(2)}$.) We shall not give the detailed proof. In view of the fact that Problem A is the solution of a problem in the calculus of variations, and that we have required C_1 and C_2 to be smooth and W to be continuous, it would be very surprising if the solution did *not* exist. Hence we would not really be justified in presenting a rather long existence proof here.

3. The behavior of $V^{(1)}$ and $V^{(2)}$ on C_1. Let us suppose that C_1 is an analytic arc or that C_1 is composed of segments of analytic arcs. We have the following:

THEOREM 1. *A segment* Γ *of* C_1 *which is an analytic arc is not a natural boundary for* $V^{(1)}$ *or* $V^{(2)}$.

This means that every point of Γ has a neighborhood such that there exists a harmonic function which is regular in the region consisting of G_1 and this neighborhood which coincides in G_1 with $V^{(1)}$. And every point of Γ has a neighborhood such that there exists a harmonic function which is regular in the region G_2 and this neighborhood and which coincides in G_2 with $V^{(2)}$.

In order to prove Theorem 1 we first solve the following problem:

PROBLEM B. Let K be a circle of radius R and center at the origin of the x, y coordinate system. Let $\bar{G}_1$ be the region interior to K lying below the x-axis and let $\bar{G}_2$ be the region interior to K lying above the x-axis. Call the part of the x-axis

interior to K, Γ. Let r, θ be polar coordinates in the x, y plane. Let $W^{(1)}(\theta)$, $-\pi \leq \theta \leq 0$, $W^{(2)}(\theta)$, $0 \leq \theta \leq \pi$ be given bounded continuous functions of θ on K satisfying $W^{(1)}(0) = W^{(2)}(0)$, $W^{(1)}(-\pi) = W^{(2)}(\pi)$, $D_1 \dfrac{\partial W^{(1)}(0)}{\partial \theta} = D_2 \dfrac{\partial W^{(2)}(0)}{\partial \theta}$, $D_1 \dfrac{\partial W^{(1)}(-\pi)}{\partial \theta} = D_2 \dfrac{\partial W^{(2)}(\pi)}{\partial \theta}$. Let it be required to find functions $\bar{V}^{(i)}$ such that

(a) $\bar{V}^{(1)}(R, \theta) = W^{(1)}(\theta), \quad -\pi \leq \theta \leq 0,$
$\bar{V}^{(2)}(R, \theta) = W^{(2)}(\theta), \quad 0 \leq \theta \leq \pi.$

(b) $\bar{V}^{(i)}$ is regular and bounded in $\bar{G}^{(i)}$.

(c) $\bar{V}^{(1)}(r, \theta) = \bar{V}^{(2)}(r, \theta)$ on Γ.

(d) $D_1 \dfrac{\partial \bar{V}^{(1)}(r, \theta)}{\partial \theta} = D_2 \dfrac{\partial \bar{V}^{(2)}(r, \theta)}{\partial \theta}$ on Γ.

Problem B can be solved by the method of circular harmonics. We obtain:

$$(1) \quad \bar{V}^{(1)}(r, \theta) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \left(\frac{r}{R}\right)^n \cos n\theta + \sum_{n=1}^{\infty} B_n \left(\frac{r}{R}\right)^n \sin n\theta, \quad -\pi \leq \theta \leq 0,$$

$$(2) \quad \bar{V}^{(2)}(r, \theta) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \left(\frac{r}{R}\right)^n \cos n\theta + \frac{D_1}{D_2} \sum_{n=1}^{\infty} B_n \left(\frac{r}{R}\right)^n \sin n\theta, \quad 0 \leq \theta \leq \pi,$$

$$(3) \quad A_m = \frac{2D_1}{(D_1 + D_2)\pi} \int_{-\pi}^{0} W^{(1)}(\theta) \cos m\theta d\theta + \frac{2D_2}{(D_1 + D_2)\pi} \int_{0}^{\pi} W^{(2)}(\theta) \cos m\theta d\theta,$$

$$(4) \quad B_m = \frac{2D_2}{(D_1 + D_2)\pi} \left[\int_{-\pi}^{0} W^{(1)}(\theta) \sin m\theta d\theta + \int_{0}^{\pi} W^{(2)}(\theta) \sin m\theta d\theta \right], \quad m = 0, 1, 2 \cdots.$$

With (3), (4) substituted into (1) and $r = R$, it is apparent that (1) represents the sum of two Fourier cosine series plus two Fourier sine series. The formal sums of the two series are easily identified. The two cosine series give us

$$\frac{D_1}{D_1 + D_2} W^1(\theta) + \frac{D_2}{D_1 + D_2} W^2(-\theta), \quad -\pi \leq \theta \leq 0.$$

The two sine series give us

$$\frac{D_2}{D_1 + D_2} W^1(\theta) - \frac{D_2}{D_1 + D_2} W^2(-\theta), \quad -\pi \leq \theta \leq 0.$$

Adding the above terms, the sum of the two cosine series plus the two sine series gives us $W^1(\theta)$, $-\pi \leq \theta \leq 0$. $W^1(\theta)$, $W^2(\theta)$ satisfy Dirichlet conditions and are

furthermore continuous within their range of definition. This means that the cosine series converge to their formal sums for $-\pi \leq \theta \leq 0$, and the sine series converge to their formal sums for $-\pi < \theta < 0$. The sine series converge to the value zero, at $\theta = 0, \theta = \pi$. The sum of the four series gives $W^1(\theta)$ for $-\pi \leq \theta \leq 0$. A similar argument shows that if $r = R$ the series on the right-hand side of (2) converges to $W^2(\theta)$ for $0 \leq \theta \leq \pi$. It is readily verified that $\bar{V}^{(1)}(r, \theta)$, $\bar{V}^{(2)}(r, \theta)$ as defined by the series in (1), (2) satisfy the remaining conditions of Problem B so that (1), (2) is the solution of Problem B. Then we have:

LEMMA 1. *Γ is not a natural boundary for $\bar{V}^{(1)}(r, \theta)$ or $\bar{V}^{(2)}(r, \theta)$ of Problem B.*

The series in (1) and (2) are each uniformly convergent for $r \leq \rho \leq R$ for all θ and hence *each* of (1), (2) defines a harmonic function *throughout* the interior of K. Q.E.D.

It is, of course, a well known theorem from function theory that under a conformal transformation a harmonic function maps into a harmonic function. We have:

LEMMA 2. *Let a neighborhood of C_1 be mapped into a neighborhood of $\bar{C}_1$, where $\bar{C}_1$ is the mapping of $\bar{C}_1$, by a conformal transformation. Let $V^{(i)}$ map into $\bar{V}^{(i)}$. Then* $D_1 \dfrac{\partial \bar{V}^{(1)}}{\partial n} = D_2 \dfrac{\partial \bar{V}^{(2)}}{\partial n}$ *on $\bar{C}_1$.*

The Lemma follows from the definition of a conformal transformation.

With the help of the solution to Problem B and Lemmas 1, 2 we can prove Theorem 1. The method of proof follows closely a proof by Bieberbach [9] of a lemma from potential theory to the effect that if V were regular in a region G, continuous in the extended domain obtained by adding to G the points of an analytic arc that belongs to the boundary of G and if V were zero on this arc, then this arc is not a natural boundary for V. We may consider $V^{(i)}$ to be the real or imaginary part of a function of a complex variable. Let the analytic arc Γ be represented in the complex plane by $z(\alpha)$, $a \leq \alpha \leq b$, where α is the real part of a complex variable $\gamma = \alpha + i\beta$. α is also chosen as the arc length along Γ.

Then the function $z(\gamma)$ maps a neighborhood K in the γ-plane of every point $\beta = 0$, $a \leq \alpha_0 \leq b$ onto a neighborhood U of the point $z = z(\alpha_0)$; this mapping is conformal. Since α represents arc length along Γ $x'(\alpha_0)^2 + y'(\alpha_0)^2 = 1$. Hence $z'(\alpha_0) \neq 0$. Hence the mapping is also simple. We choose for K a circle with center on the real axis at α_0. This determines a region U in the z-plane. Let G_i and $V^{(i)}$, $i = 1, 2$, refer to the regions and functions of Problem A. From the intersection of G_i and the boundary of U we obtain the values of $V^{(i)}$ on the boundary of U. By the inverse of the mapping mentioned the boundary values on U are transformed to boundary values on K. These satisfy the conditions for Problem B. From the results of Problem B each of the transforms of the functions $V^{(i)}$ to K is analytic in K, and hence by an inverse transformation each of the $V^{(i)}$ is analytic throughout U. Q.E.D.

The domain of analyticity of the functions continued across Γ is difficult to determine. It will depend on the location of the zeros and poles of $z(\gamma)$, as well as on the location of non-analytic points of C_1, and the distance of C_1 to C_2, and also the location of the mapping of C_2.

4. The connection between power series for $\bar{V}^{(1)}$ and $\bar{V}^{(2)}$. Let O be the center of the circle K. Let t and n be the abscissa and ordinate measured from O. We

label the value of $\bar{V}^{(i)}$ at O, $\bar{V}_0^{(i)}$ and we let $\bar{V}_t^{(i)}$, $\bar{V}_n^{(i)}$, $\bar{V}_{tt}^{(i)}$, etc., be partial derivatives of $\bar{V}^{(i)}$ with respect to t and n at O. We may expand $\bar{V}^{(i)}$ about O in power series convergent in K:

$$\bar{V}^{(1)} = \bar{V}_0^{(1)} + \bar{V}_n^{(1)}n + \bar{V}_t^{(1)}t + \tfrac{1}{2}\bar{V}_{nn}^{(1)}n^2 + \bar{V}_{nt}^{(1)}nt + \tfrac{1}{2}\bar{V}_{tt}^{(1)}t^2 + \cdots, \tag{5}$$

$$\bar{V}^{(2)} = \bar{V}_0^{(2)} + \bar{V}_n^{(2)}n + \bar{V}_t^{(2)}t + \tfrac{1}{2}\bar{V}_{nn}^{(2)}n^2 + \bar{V}_{nt}^{(2)}nt + \tfrac{1}{2}\bar{V}_{tt}^{(2)}t^2 + \cdots, \tag{6}$$

THEOREM 2. *If the power series for $\bar{V}^{(1)}$ is given the coefficients in the power series for $\bar{V}^{(2)}$ may be determined.*

Since $\bar{V}^{(2)} = \bar{V}^{(1)}$ on Γ it follows that

$$\bar{V}_t^{(2)} = \bar{V}_t^{(1)}, \quad \bar{V}_{tt}^{(2)} = \bar{V}_{tt}^{(1)}. \tag{7}$$

Since $D_2\bar{V}_n^{(2)} = D_1\bar{V}_n^{(1)}$, it follows that

$$D_2\bar{V}_{nt}^{(2)} = D_1\bar{V}_{nt}^{(1)} \tag{8}$$

Since $\bar{V}_{tt}^{(2)} = \bar{V}_{tt}^{(1)}$, $\bar{V}_{tt}^{(i)} + \bar{V}_{nn}^{(i)} = 0$, $i = 1, 2$, it follows that $\bar{V}_{nn}^{(2)} = \bar{V}_{nn}^{(1)}$. We have expressed $\bar{V}^{(2)}$ and all its derivatives through second order in terms of $\bar{V}^{(1)}$ and its derivatives through second order. Let $\xi = D_1/D_2$. We have

$$\bar{V}^{(2)} = \bar{V}_0^{(1)} + \xi\bar{V}_n^{(1)}n + \bar{V}_t^{(1)}t + \tfrac{1}{2}\bar{V}_{nn}^{(1)}n^2 + \xi\bar{V}_{nt}^{(1)}nt + \tfrac{1}{2}\bar{V}_{tt}t^2. \tag{9}$$

We shall be concerned with terms through second order only. However, one may show that (7), (8) and the derivatives of the equations $V_{tt}^{(i)} + V_{nn}^{(i)} = 0$ always give enough equations so that the derivatives of $\bar{V}^{(2)}$ can be expressed in terms of those of $\bar{V}^{(1)}$.

5. Five point algebraic approximations. We consider first the case in which Γ is, locally, a straight line. We suppose that Γ passes through a meshpoint making an angle α with the x-axis. Let the meshpoint be labeled "a" and let the four nearest neighbors be labeled "b," "c," "d," "e" as shown in figure 1. From Theorem 1 it follows that $V^{(i)}$ should satisfy Laplace's equation at "a." Let the

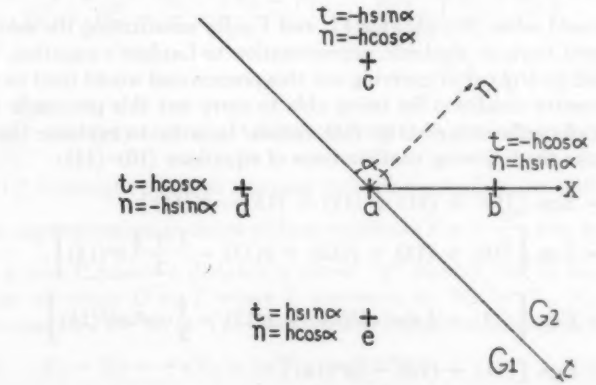


FIGURE 1.

network values of $V^{(i)}$ at a, b, c, d, e be V_a, V_b, V_c, V_d, V_e. It is not necessary to superscript these quantities since there is only one function value associated with each meshpoint. Let h be the mesh spacing. We have the following theorem:

THEOREM 3. *When $\xi \neq 1$, except for $\alpha = \dfrac{n\pi}{4}$, $n = 0, 1, \cdots, 7$, there exists no algebraic approximation for Laplace's equation in terms of V_a, V_b, V_c, V_d, V_e which is such that the local truncation error tends to zero as $h \to 0$.*

For definiteness, assume that $\pi/2 \leq \alpha \leq \pi$. Then meshpoints b and c are on the boundary of or inside G_2 and d and e are on the boundary of or inside G_1.

We choose an origin at "a" and introduce coordinates t, n where n is the normal at "a" from G_1 into G_2 and t is the distance along Γ from a. The direction of positive t is chosen so that the t, n coordinate system is right-handed. It is assumed that a circle about a which just encloses b, c, d, e has radius less than the circle of convergence of the power series (5), (6) and that Γ is straight inside this circle. Since Γ is straight it is not necessary to distinguish between Γ, $\bar{\Gamma}$; $V^{(i)}$, $\bar{V}^{(i)}$, etc. The values assumed by t and n at b, c, d, e are given in figure 1. Partial derivatives V_n, V_t, etc., refer to $V_n^{(1)}$, $V_t^{(1)}$, etc., in what follows. Making use of (5), (9) we have

$$V_b - V_a = \xi h \sin\alpha V_n - h\cos\alpha V_t + \tfrac{1}{2}h^2\sin^2\alpha V_{nn} - \xi h^2 \sin\alpha\cos\alpha V_{nt} + \tfrac{1}{2}h^2\cos^2\alpha V_{tt} + O(h^3), \tag{10}$$

$$V_c - V_b = -\xi h\cos\alpha V_n - h\sin\alpha V_t + \tfrac{1}{2}h^2\cos^2\alpha V_{nn} + \xi h^2\sin\alpha\cos\alpha V_{nt} + \tfrac{1}{2}h^2\sin^2\alpha V_{tt} + O(h^3), \tag{11}$$

$$V_d - V_a = -h\sin\alpha V_n + h\cos\alpha V_t + \tfrac{1}{2}h^2\sin^2\alpha V_{nn} - h^2\sin\alpha\cos\alpha V_{nt} + \tfrac{1}{2}h^2\cos^2\alpha V_{tt} + O(h^3), \tag{12}$$

$$V_e - V_a = h\cos\alpha V_n + h\sin\alpha V_t + \tfrac{1}{2}h^2\cos^2\alpha V_{nn} + h^2\sin\alpha\cos\alpha V_{nt} + \tfrac{1}{2}h^2\sin^2\alpha V_{tt} + O(h^3), \tag{13}$$

$$0 = V_{nn} + V_{tt}. \tag{14}$$

Suppose we could solve (10)–(13) for V_{nn} and V_{tt}. By substituting the solution into (14) we would have an algebraic approximation to Laplace's equation. The $O(h^3)$ terms would be $O(h)$ after carrying out this process and would tend to zero as $h \to 0$. A necessary condition for being able to carry out this process is that the determinant of coefficients of (10)–(14) vanish. In order to evaluate the determinant, we take the following combinations of equations (10)–(14):

$$\text{Eq. 15} = \text{Eqs.}\,[(10) + (11) + (12) + (13) - h^2(14)],$$

$$\text{Eq. 16} = \text{Eqs.}\left[(10) + (11) + \xi(12) + \xi(13) - \frac{1+\xi}{2}h^2(14)\right],$$

$$\text{Eq. 17} = \text{Eqs.}\left[(11) - \tfrac{1}{2}\sin^2\alpha h^2(14) + \xi(12) - \frac{\xi}{2}\cos^2\alpha h^2(14)\right],$$

$$\text{Eq. 18} = \text{Eqs.}\,[(11) + (12) - \tfrac{1}{2}h^2(14)],$$

$$\text{Eq. 19} = \text{Eq. (14)}.$$

The transformation of (11)–(14) to (15)–(19) is non-singular for $\xi \neq 1$. We obtain

$$(15)\quad (\xi-1)(\sin\alpha-\cos\alpha)hV_n = V_b + V_c + V_d + V_e - 4V_a + O(h^3),$$

$$(16)\quad (\xi-1)(\sin\alpha+\cos\alpha)hV_t = V_b + V_c - 2V_a + \xi(V_d + V_e - 2V_a) + O(h^3),$$

$$(17)\quad -\xi(\sin\alpha+\cos\alpha)hV_n + (\xi\cos\alpha-\sin\alpha)hV_t + \frac{1-\xi}{2}(\cos^2\alpha-\sin^2\alpha)h^2V_{nn} = V_e - V_a + \xi(V_d - V_a) + O(h^3),$$

$$(18)\quad -(\sin\alpha+\xi\cos\alpha)hV_n + (\cos\alpha-\sin\alpha)hV_t + (\xi-1)\sin\alpha\cos\alpha h^2V_{nt} = V_c + V_d - 2V_a + O(h^3),$$

$$(19)\quad V_{nn} + V_{tt} = 0.$$

The equation system (15)–(19) is triangular. Its determinant D is

$$(20)\qquad D = \tfrac{1}{2}(\xi-1)^4(\sin^2\alpha-\cos^2\alpha)^2\sin\alpha\cos\alpha h^6.$$

D is zero for $\xi \neq 1$, $\frac{\pi}{2} \leq \alpha \leq \pi$ only when $\alpha = \frac{\pi}{2}$, $\alpha = \frac{3\pi}{4}$, $\alpha = \pi$. By symmetry, D would be zero also only for $\alpha = \frac{n\pi}{4}$, $n = 0, 1, \cdots, 7$ for $0 \leq \alpha < 2\pi$. This proves Theorem 3. The algebraic approximations which we get, for $\alpha = \frac{3\pi}{4}$ and $\alpha = \pi$ are, respectively:

$$(21)\quad V_{tt}^{(i)} + V_{nn}^{(i)} = \frac{2D_2}{D_1+D_2}\left[\frac{V_b - 2V_a + V_c}{h^2}\right] + \frac{2D_1}{D_1+D_2}\left[\frac{V_d - 2V_a + V_e}{h^2}\right] + O(h) = 0$$

$$(22)\quad V_{tt}^{(i)} + V_{nn}^{(i)} = \frac{V_b - 2V_a + V_d}{h^2} + \frac{2D_2}{D_1+D_2}\left(\frac{V_c - V_a}{h^2}\right) + \frac{2D_1}{D_1+D_2}\left(\frac{V_e - V_a}{h^2}\right) + O(h) = 0$$

(21) and (22) may also be obtained by an application of Gauss's theorem [6, 7].

If Γ is straight but does not pass through meshpoints, we still obtain an algebraic approximation in terms of four neighbors if $\alpha = \frac{n\pi}{2}$. For example, suppose $\alpha = \pi$ and Γ passes a distance η above "a" leaving "a" in G_1, "c" in G_2. We choose an origin O on Γ where Γ intersects ac. We let V_t, V_n, etc., represent derivatives of $V^{(1)}$ at O. The continuation equations analogous to (10)–(13) are

$$(23)\quad V_a - V_0 = -\eta V_n + \tfrac{1}{2}\eta^2 V_{nn} + O(h^3),$$

$$(24)\quad V_b - V_0 = -\eta V_n + hV_t + \tfrac{1}{2}\eta^2 V_{nn} - h\eta V_{nt} + \tfrac{1}{2}h^2 V_{tt} + O(h^3),$$

$$V_c - V_0 = \xi(h-\eta)V_n + \tfrac{1}{2}(h-\eta)^2 V_{nn} + O(h^3), \tag{25}$$

$$V_d - V_0 = -\eta V_n - hV_t + \tfrac{1}{2}\eta^2 V_{nn} + h\eta V_{nt} + \tfrac{1}{2}h^2 V_{tt} + O(h^3), \tag{26}$$

$$V_e - V_0 = -(\eta + h)V_n + \tfrac{1}{2}(h+\eta)^2 V_{nn} + O(h^3). \tag{27}$$

Subtracting (23) from (25) we obtain

$$V_c - V_a = [\xi(h-\eta) + \eta]V_n + \tfrac{1}{2}(h^2 - 2h\eta)V_{nn} + O(h^3). \tag{28}$$

Subtracting (27) from (23)

$$V_a - V_e = hV_n - \tfrac{1}{2}(2\eta h + h^2)V_{nn} + O(h^3). \tag{29}$$

Dividing (28) by $\xi(h-\eta) + \eta$ and subtracting (29) divided by h, we obtain

$$\frac{V_c - V_a}{\xi(h-\eta)+\eta} - \frac{V_a - V_e}{h} = \frac{1}{2}\left[\frac{h^2 - 2h\eta}{\xi(h-\eta)+\eta} + h + 2\eta\right]V_{nn} + O(h^2). \tag{30}$$

Taking (24) minus twice (23) plus (26) and then dividing by h^2, we have

$$\frac{V_b - 2V_a + V_d}{h^2} = V_{tt} + O(h). \tag{31}$$

Dividing (30) by the factor multiplying V_{nn} in (30), then adding equation (31) and making use of $V_{nn} + V_{tt} = 0$, we obtain

$$V_{nn} + V_{tt} = 2\left[\frac{h^2 - 2h\eta}{\xi(h-\eta)+\eta} + h + 2\eta\right]^{-1}\left[\frac{V_c - V_a}{\xi(h-\eta)+\eta} - \frac{V_a - V_e}{h}\right] + \frac{V_b - 2V_a + V_d}{h^2} + O(h) = 0. \tag{32}$$

Equation (32) is our algebraic approximation in this case. It reduces to (22) when $\eta = 0$. If we accept the idea that $V^{(4)}$ be continuable to neighboring meshpoints with terms through second order in a Taylor Series, as is implied by equations (23)–(27), then equation (32) is the unique algebraic representation of our problem for this case. Terms through second order are necessary in order that the local truncation error vanish as $h \to 0$. Some schemes which appear on the surface to represent our problem with local truncation error tending to zero when $h \to 0$ do not do so. Douglas, Garder, and Peaceman treated a moving boundary problem. According to Douglas, Garder, and Peaceman [10] to treat the case just considered we introduce an additional meshpoint at 0. Then we can easily develop a divided difference approximation to Laplace's equation at a in terms of V_a, V_b, V_0, V_d, V_e. We obtain

$$V_{nn} + V_{tt} = \frac{2}{h+\eta}\left[\frac{V_0 - V_a}{\eta} - \frac{V_a - V_e}{h}\right] + \frac{V_b - 2V_a + V_d}{h^2} + O(h) = 0. \tag{33}$$

However, we now have an extra unknown V_0 so that we must have an additional equation. As our additional equation we choose an algebraic approximation to

condition (d) of Problem A, namely

$$(34)\quad D_2\frac{V_c - V_0}{h-\eta} = D_1\frac{V_0 - V_a}{\eta} - \left[\frac{D_1}{2}\eta + \frac{D_2}{2}(h-\eta)\right]V_{nn} + O(h^2).$$

The truncation error term which is $O(h)$ is explicitly given in (34). We can compare (33), (34) with (32) by using (34) to eliminate V_0 from (33). Solving (34) for $\frac{V_0}{\eta}$ and then subtracting $\frac{V_a}{\eta}$ from both sides, we obtain

$$(35)\quad \frac{V_0 - V_a}{\eta} = \frac{V_c - V_a}{\xi(h-\eta)+\eta} + \frac{1}{2}\left[\frac{h-\eta+\xi\eta}{\xi(h-\eta)+\eta}\right](h-\eta)V_{nn} + O(h^2).$$

Substituting (35) into (33), we obtain

$$(36)\quad V_{nn} + V_{tt} = \frac{2}{h+\eta}\left[\frac{V_c - V_a}{\xi(h-\eta)+\eta} - \frac{V_a - V_b}{h}\right] + \frac{V_b - 2V_a + V_d}{h^2} + \left[\frac{(h-\eta)+\xi\eta}{\eta+\xi(h-\eta)}\frac{h-\eta}{h+\eta}\right]V_{nn} + O(h) = 0.$$

Equation (36) does not agree with (32) unless $\eta = \frac{h}{1-\xi}$ or $\eta = h$. Furthermore, in general, the local truncation error does not vanish as $h \to 0$. When $D_1 = D_2$, it is easily shown that $V^{(1)}$ and $V^{(2)}$ are parts of the same harmonic function. (This is a theorem of potential theory, cf. Kellogg [11]). Since this is so, it would be expected that (32) and (36) should reduce to the ordinary five point formula for approximating the Laplace equation. (32) does this but (36) does not. The local truncation error in (36) remains finite when $h \to 0$ even for $\xi = 1$. The fact that the *local* truncation error in (36) remains finite as $h \to 0$ does not mean that the error in the solution remains finite as $h \to 0$. The local truncation error may be regarded as a ficticious charge density distribution and the error in the solution as the potential of this distribution. As $h \to 0$ the local truncation error in (36) remains finite. Hence the error in the solution approaches the potential of a charged layer whose thickness approaches zero while the charge density in the layer remains finite. The potential of such a distribution will become everywhere arbitrarily small as the layer thickness goes to zero.

Equation (34) causes difficulties when the system of algebraic approximations is to be solved by an iteration method. The rate of convergence of most methods will tend to zero as $\eta \to 0$. Of course, the local truncation error in (36) may be reduced by approximating V_{nn} by a finite difference approximation.

6. Algebraic approximations with more than five meshpoints. Except for special cases as we have discussed, no algebraic approximation with local truncation error which vanishes as $h \to 0$ is possible for the points adjacent to Γ in terms of five meshpoints. If we make use of one additional neighbor and annex one additional equation to (10)–(14), then there is no difficulty in obtaining an algebraic approximation. We solve (10)–(13) together with our extra equation to obtain V_{nn} and V_{tt} in terms of values of V at meshpoints. Then we substitute these expressions for V_{nn} and V_{tt} into equation (14).

But which extra neighbor do we choose, and why? The problem becomes very unsymmetrical. If we make use of several additional neighbors, say the four corner points in figure 2, then a symmetrical formulation is possible, but now the solution is no longer unique, there being more equations than necessary.

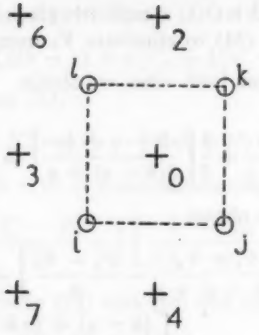


FIGURE 2.

In figure 2 we show the four neighbors of figure 1 relabeled "1," "2," "3," "4." We show the central meshpoint "a" relabeled "0," and we show the corner points "5," "6," "7," "8." Let us suppose that the discontinuity intersects one of the line segments 0–1, 0–2, 0–3, 0–4, so that a special representation is necessary. We choose an arbitrary origin on the discontinuity and let the coordinates of the meshpoints relative to this origin be (t_i, n_i), $i = 0, 1, \cdots, 8$. Then the continuation equations analogous to (10)–(14) are

$$(37)\quad V_i - V_0 = (\xi^{\delta_i} n_i - \xi^{\delta_0} n_0) V_n + (t_i - t_0) V_t + \tfrac{1}{2}(n_i^2 - n_0^2) V_{nn} + (\xi^{\delta_i} n_i t_i - \xi^{\delta_0} n_0 t_0) V_{nt} + \tfrac{1}{2}(t_i^2 - t_0^2) V_{tt}, \quad i = 1, 2, 3, 4.$$

where $\delta_i = 0$ when meshpoint i is in G_1, $\delta_i = 1$ when meshpoint i is in G_2.

In addition we have the continuation equations for the corner points,

$$(38)\quad V_i - V_0 = (\xi^{\delta_i} n_i - \xi^{\delta_0} n_0) V_n + (t_i - t_0) V_t + \tfrac{1}{2}(n_i^2 - n_0^2) V_{nn} + (\xi^{\delta_i} n_i t_i - \xi^{\delta_0} n_0 t_0) V_{nt} + \tfrac{1}{2}(t_i^2 - t_0^2) V_{tt}, \quad i = 5, 6, 7, 8.$$

The values of V_{nn} and V_{tt} obtained from any solution of (37), (38) are independent of where the origin is chosen on the discontinuity. To see this, suppose that the origin is displaced on the discontinuity so that t_i is replaced by $t_i + \Delta t$. Then the coefficients of V_n, V_t and V_{nn} in (37), (38) are unchanged, but the coefficients of V_{nt}, V_{tt} are increased by $(\xi^{\delta_i} n_i - \xi^{\delta_0} n_0)\Delta t$ and $(t_i - t_0)\Delta t$ respectively.

These latter quantities are proportional to the coefficients of V_n and V_t. When we solve equations (37), (38) for V_{nn} and V_{tt}, $(\xi^{\delta_i} n^i - \xi^{\delta_0} n_0) V_n$ and $(t_i - t_0) V_t$ must be eliminated, and when this is done the extra terms $(\xi^{\delta_i} n_i - \xi^{\delta_0} n_0)\Delta t V_{nt}$ and $(t_i - t_0)\Delta t V_{tt}$ will also drop out of the equations.

In order to obtain a non-singular set of 5 equations from (37), (38), we may choose the set of four equations in (37) and as a fifth equation choose

$$(39)\quad V_5 - V_6 + V_7 - V_8 = a_1 V_n + a_2 V_t + a_3 V_{nn} + a_4 V_{nt} + a_5 V_{tt},$$

where a_1, a_2, a_3, a_4, a_5 are the coefficients of V_n, V_t, V_{nn}, V_{nt}, V_{tt} obtained when we form equation (39) from (38), by taking the combination of equations (38) indicated by the left side of (39). The system of equations (37), (39) is a plausible set to use for obtaining an algebraic representation. Loosely speaking, (37) does not give a "well-set" determination of V_{nt} while (39) does. $\left(V_{nt}\right.$ drops out of (37) when $\alpha = \dfrac{n\pi}{2}\cdot\Big)$

From the system (37), (39) we obtain (21), (22), and (32) in the respective limiting cases. Arms, Gates, and Zondek [12] have shown that extrapolated line relaxation may be used to solve difference equations involving nine-point formulas. Of course, it is not practical to solve (37), (39) to obtain a explicit formula analogous to (21), (22), (32) for the general case. However, it is not difficult to use a computer to calculate the coefficients in the algebraic representation for any given case. First we compute the matrix of coefficients of equations (37), (39) and then invert this matrix, and finally we combine elements of the inverse to obtain the values of the coefficients which multiply each V_i in the algebraic representation of $V_{nn} + V_{tt} = 0$.

Other methods for determining an algebraic approximation may be based on an integral formulation of our problem. Referring to figure 2, we let

$$E_{lk} = \int_l^k D_m V_y^{(m)} dx, \quad \bar{E}_{jk} = \int_j^k D_m V_x^{(m)} dy, \tag{40}$$

i.e., E_{lk} is the line integral of $D_m V_y^{(m)}$ along the line lk shown in figure 2, $\bar{E}_{jk}$ the line integral from j to k. Superscript m will be one over a segment of the line of integration lying in region one, and will be two over a segment in region two. Then as may be shown from the conditions of Problem A, we have

$$E_{lk} + \bar{E}_{jk} - \bar{E}_{il} - E_{ij} = 0. \tag{41}$$

The integrals E_{lk}, $\bar{E}_{jk}$, etc. may be approximated by algebraic approximations. Again values of V at some of the corner points 5, 6, 7, 8 will be required in making the approximations. Algebraic approximations based on (40), (41), have the advantage that the resulting equations obey a conservation law which is an algebraic analogue of Gauss's theorem $\oint D_m \nabla V^{(m)} d\mathbf{S} = 0$. In the case of moving boundary problems this method also has the advantage of giving a unique velocity for the moving boundary.

7. Curved Interfaces. If Γ is an arc with continuous curvature, it may be approximated in neighborhoods by simple analytic arcs, for example, by osculating circles. The mesh spacing must be small enough so that the radial distances from the osculating circle to the nine meshpoints to be used in the algebraic approximation are less than and are preferably small compared to the radius of curvature. There are two ways in which we may obtain the continuation equations. One method consists in mapping the osculating circle onto a segment of the real axis in the γ-plane. This mapping is easily obtained. The mapping $Z = Z_0 + Re^{\beta/R} e^{i(\alpha/R)}$ maps the real axis of the $\gamma = \alpha + i\beta$ plane for $0 \leq \alpha \leq 2\pi$ into a circle with center at Z_0 and radius R in the Z-plane. The inverse of this mapping will map a circle

onto the segment $(0, 2\pi)$ of the real axis. Under the mapping the nine meshpoints to be used in the algebraic approximation are mapped into nine meshpoints in the γ plane. The coordinates of these meshpoints in the γ plane, relative to an origin on the real axis, are substituted into the continuation equations (37), (38).

A second method consists in expanding $V^{(i)}$ in a power series about a point O on the osculating circle, and choosing t to be arc length from O on the osculating circle. We have as before $V_n^{(2)} = \xi V_n^{(1)}$, $V_t^{(2)} = V_t^{(1)}$, $V_{tt}^{(2)} = V_{tt}^{(1)}$, $V_{nt}^{(2)} = \xi V_{nt}^{(1)}$ but $V_{nn}^{(2)} \neq V_{nn}^{(1)}$. Expressing Laplace's equation in polar coordinates with origin at the center of curvature, we have

$$V_{nn}^{(i)} + \frac{1}{r} V_n^{(i)} + V_{tt}^{(i)} = 0 \tag{42}$$

whence it follows that

$$V_{nn}^{(2)} = V_{nn}^{(1)} + \frac{1 - \xi}{r} V_n^{(1)}. \tag{43}$$

Then the continuation equations (37), (38) become

$$\begin{aligned} V_i - V_0 = {} & \left[\left(\xi^{3/2} n_i + \delta_i \frac{1 - \xi}{r} \frac{n_i^2}{2}\right) - \left(\xi^{3/2} n_0 + \delta_0 \frac{1 - \xi}{r} \frac{n_0^2}{2}\right)\right] V_n \\ & + (t_i - t_0) V_t + \tfrac{1}{2}(n_i^2 - n_0^2) V_{nn} + (\xi^{3/2} n_i t_i - \xi^{3/2} n_0 t_0) V_{nt} \\ & + \tfrac{1}{2}(t_i^2 - t_0^2) V_{tt}, \quad i = 1, 2, 3, \cdots, 8. \end{aligned} \tag{44}$$

In (42), (43), (44) r is positive if the center of curvature is in G_1, negative if in G_2. This follows because n was the normal from G_1 into G_2.

8. Other equations. It can be shown that the power series method here described may be used to obtain algebraic approximations to analogues of Problem A for certain other elliptic or parabolic equations, for example, for the Helmholtz equation and for the diffusion equation.

Computer Usage Company, Inc.
18 East 41st Street
New York, New York

1. J. H. JEANS, *The Mathematical Theory of Electricity and Magnesism*, Cambridge University Press, London, 1948.
2. W. R. SMYTHE, *Static and Dynamic Electricity*, McGraw-Hill Book Co., New York, 1939.
3. R. E. KIDDER, "Motion of an interface between two immiscible liquids of unequal density in a porous solid," *Jn.* of *Appl. Phys.*, v. 27, 1956, p. 1546–1548.
4. G. H. SHORTLEY & R. WELLER, "The numerical solution of Laplace's equation," *Jn. Appl. Phys.*, v. 9, 1938, p. 334–348.
5. R. V. VISWANATHAN, "Solution of Poisson's equation by relaxation method-normal gradient specified on curved boundaries," *MTAC*, v. XI, 1957, p. 67–78.
6. R. S. VARGA, "Numerical solution of the two-group diffusion equations in $x - y$ geometry," WAPD-159 AEC Research and Development Report, 1956, p. 8–12.
7. E. L. WACHSPRESS, "CURE: A generalized two-space-dimension multigroup coding for the IBM 704," KAPL-1724 AEC Research and Development Report, 1957, p. 20–28.
8. O. D. KELLOGG, *Foundations of Potential Theory*, Frederick Unger, New York, 1929, p. 286–315.
9. L. BIEBERBACH, *Conformal Mapping*, Chelsea Publishing Co., New York, 1953, p. 149–151.
10. J. DOUGLAS, JR., A. O. GARDNER, & D. W. PEACEMAN, "Numerical solution of a two-dimensional moving boundary problem," Paper presented at Association for Computing Machinery Meeting, University of Houston, June 19–21, 1957. (Unpublished.)
11. O. D. KELLOGG, Op. Cit., p. 261.
12. L. J. ARMS, L. D. GATES, & B. ZONDEK, "A method of block iteration," Soc. for Industrial and Applied Math., *Jn.*, v. 4, 1956, p. 220–229.

Simplified Calculation of the Exponential Integral

By James Miller and R. P. Hurst

Calculations of the negative exponential integral function [1]

$$-\operatorname{Ei}(-x) = \int_x^\infty \frac{e^{-t}}{t}\,dt \tag{1}$$

are usually made using one method for $x < 3$, another for $3 < x < 20$, and a third for $x > 20$. The intermediate region is of considerable interest for many applications, and it is here that the function is most difficult to compute. The Ruedenberg [2] and Kotani [3] methods are valid for any argument, but difficulties in application limit their usefulness to the intermediate region where less complicated methods fail.

The authors wish to present a simple method which is useful over the entire range of arguments. The method is based on a Taylor's expansion of the slowly varying *exponential product function*, $-e^x \operatorname{Ei}(-x)$. An interpolation formula is obtained by which this function may be computed from a table value, $-e^{x_0} \operatorname{Ei}(-x_0)$; then $-\operatorname{Ei}(-x)$ is obtained upon dividing by the exponential. The accuracy of the result is limited only by the accuracy of the table value and that of the computed exponential. The positive exponential integral

$$\operatorname{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t}\,dt$$

may be computed by analogous means.

The formulae for the negative exponential product function are developed as follows:

Let

$$g(x) = -e^x \operatorname{Ei}(-x), \tag{2}$$

then

$$\begin{aligned} g'(x) &= g(x) - x^{-1} \\ g''(x) &= g'(x) + x^{-2} \\ g'''(x) &= g''(x) - 2!x^{-3} \\ g^{iv}(x) &= g'''(x) + 3!x^{-4}. \end{aligned}$$

One may show by induction that

$$g^{(n)}(x) = g^{(n-1)}(x) + (-1)^n(n-1)!x^{-n}. \tag{3}$$

The Taylor's expansion, then, is

$$\begin{aligned} -e^x \operatorname{Ei}(-x) = {} & g(x_0) + g'(x_0)(x - x_0) + (1/2!)g''(x_0)(x - x_0)^2 \\ & + (1/3!)g'''(x_0)(x - x_0)^3 + \cdots + (1/n!)g^{(n)}(x_0)(x - x_0)^n + R_{n+1} \end{aligned} \tag{4}$$

Received 20, September 1957. This research was supported by the Robert A. Welch Foundation of Houston and the Office of Ordnance Research, U. S. Army.

where

$$R_{n+1} = \frac{1}{(n+1)!} g^{(n+1)}(x_1)(x - x_0)^{n+1}$$

with $x < x_1 < x_0$.

The preceding equations may be combined to produce the simple formula

$$-e^x \operatorname{Ei}(-x) = T_0 + \sum_{n=1}^{\infty} T_n \tag{5}$$

where

$$T_0 = g(x_0) = -e^{x_0} \operatorname{Ei}(-x_0) \tag{6}$$

and the T's are related by the recursion relation

$$T_n = \frac{x - x_0}{n}\left[T_{n-1} + \frac{(x - x_0)^{n-1}}{(-x_0)^n}\right] \quad (n \geq 1). \tag{7}$$

For the positive exponential product function we have

$$e^{-x} \operatorname{Ei}(x) = U_0 + \sum_{n=1}^{\infty} U_n \tag{8}$$

where

$$U_0 = e^{-x_0} \operatorname{Ei}(x_0) \tag{9}$$

and

$$U_n = -\frac{x - x_0}{n}\left[U_{n-1} + \frac{(x - x_0)^{n-1}}{(-x_0)^n}\right] \quad (n \geq 1). \tag{10}$$

Equations (5) through (10) are very convenient for computing the negative or positive exponential product functions where a table value is known. For example, with $\left|\frac{x - x_0}{x_0}\right| \leq \frac{1}{50}$, convergence to 16 figures is obtained with 12 or fewer terms; convergence to 8 figures requires 6 or fewer terms. The spacing of the arguments in the accompanying table is such that this condition is fulfilled over most of the range $2.5 \leq x \leq 80$.

To prevent loss of figures, $x - x_0$ should be negative for computing $-e^x \operatorname{Ei}(-x)$ and positive for computing $e^{-x} \operatorname{Ei}(x)$. The accompanying table of $-e^x \operatorname{Ei}(-x)$ was computed starting with the value for $x = 80$, where $-e^x \operatorname{Ei}(-x)$ was computed using the asymptotic expansion; each computed value became the T_0 for the next lower argument. The table of $e^{-x} \operatorname{Ei}(x)$ was computed going up from the value of the function for $x = 0.2$, where $\operatorname{Ei}(x)$ was computed using the usual MacLaurins series, then multiplied by e^{-x} to obtain the product function. All calculations were made on an IBM 650 computer using double precision, floating point arithmetic.

x

0.20
0.25
0.30
0.35
0.40
0.45
0.50
0.55
0.60
0.65
0.70
0.75
0.80
0.85
0.90
0.95
1.00
1.05
1.10
1.15
1.20
1.25
1.30
1.35
1.40
1.45
1.50
1.55
1.60
1.65
1.70
1.75
1.80
1.85
1.90
1.95
2.00
2.05
2.10
2.15
2.20
2.25
2.30
2.35
2.40
2.45
2.50
2.55
2.60
2.65
2.70
2.75
2.80
2.85
2.90

EXPONENTIAL INTEGRALS AND EXPONENTIAL PRODUCT FUNCTIONS

x	$e^{-x}E_1(x)$	$E_1(x)$	$-e^{x}E_1(-x)$	$-E_1(-x)$
0.20	-0.6728006649831373(0)	-0.8217605879024003(0)	0.1493348746932240(1)	0.1222650544183893(1)
0.25	-0.4225331193688149(0)	-0.5425432646619137(0)	0.1340885444831393(1)	0.1044282634443738(1)
0.30	-0.2242223687152438(0)	-0.3026685392658259(0)	0.1222535605080586(1)	0.9056766516758467(0)
0.35	-0.6302307596780976(-1)	-0.8943400191844643(-1)	0.1127045349862318(1)	0.7942154346208358(0)
0.40	0.7022622616783961(-1)	0.1047652186193248(0)	0.1047828008456006(1)	0.7023801188656624(0)
0.45	0.1816318255906013(0)	0.2848554053465651(0)	0.9807147233583903(0)	0.6253313163232692(0)
0.50	0.2754982985512703(0)	0.4542199048631736(0)	0.9229106324837305(0)	0.5597735947761608(0)
0.55	0.3549918279209135(0)	0.6152906570621864(0)	0.8724573131594157(0)	0.5033640813923938(0)
0.60	0.4225198103287018(0)	0.7698812899373594(0)	0.8279334352735088(0)	0.4563795031894021(0)
0.65	0.4799618229587596(0)	0.9193866482454431(0)	0.7882775692635576(0)	0.4115169759494795(0)
0.70	0.5288172627521610(0)	0.1064907194624291(1)	0.7526780200295871(0)	0.3737688432335091(0)
0.75	0.5703036403055990(0)	0.1207332816001222(1)	0.7205015065870451(0)	0.3403408129112300(0)
0.80	0.6054242952633644(0)	0.1347396548212326(1)	0.6912453978028315(0)	0.3105965785455430(0)
0.85	0.6350164235265754(0)	0.1485714176225254(1)	0.6645047874381068(0)	0.2840192685024615(0)
0.90	0.6597860062607939(0)	0.1622811713696867(1)	0.6399492266392997(0)	0.2601839393259996(0)
0.95	0.6803337743392053(0)	0.1759145611867690(1)	0.6173059209526070(0)	0.2387375236537346(0)
1.00	0.6971748832350661(0)	0.1895117816355937(1)	0.5963473623231941(0)	0.2193839343955203(0)
1.05	0.7107540774523513(0)	0.2031087184099664(1)	0.5768820704052510(0)	0.2018728132201966(0)
1.10	0.7214575567019491(0)	0.2167378279563403(1)	0.5587475561702364(0)	0.1859090045360402(0)
1.15	0.7296223877856724(0)	0.2304288251855629(1)	0.5418049014535823(0)	0.1715553536299986(0)
1.20	0.7355440612554694(0)	0.2442092285192652(1)	0.5259345318947846(0)	0.1584084368514626(0)
1.25	0.7394826255791207(0)	0.2581047974355476(1)	0.5110328836740476(0)	0.1464133725259102(0)
1.30	0.7416677163903875(0)	0.2721398880232024(1)	0.4970097482616443(0)	0.1354509578491291(0)
1.35	0.7423027172881568(0)	0.2863377453173074(1)	0.4837861375242837(0)	0.1254168443886644(0)
1.40	0.7415682306020866(0)	0.3007207464150644(1)	0.4712925524860814(0)	0.1162193125713579(0)
1.45	0.7396249943951793(0)	0.3153106049387744(1)	0.4594675683131685(0)	0.1077774398689766(0)
1.50	0.7366163509600129(0)	0.3301285449129798(1)	0.4482566629415830(0)	0.1000195824066327(0)
1.55	0.7326703489608585(0)	0.3451954502797438(1)	0.4376112831118401(0)	0.9288210816420927(-1)
1.60	0.7279015439636994(0)	0.3605319949019469(1)	0.4274879752994281(0)	0.8630833369753978(-1)
1.65	0.7224125488366733(0)	0.3761587560694135(1)	0.4178477732671577(0)	0.8024762667334322(-1)
1.70	0.7162953753046061(0)	0.3920963201354904(1)	0.4086555960005607(0)	0.7465464440125304(-1)
1.75	0.7096326000236425(0)	0.4083653659076957(1)	0.3998797703779726(0)	0.6948868460463879(-1)
1.80	0.7024983823417449(0)	0.4249867557487934(1)	0.3914916189703562(0)	0.6471312936386886(-1)
1.85	0.6949593560157989(0)	0.4419816079804075(1)	0.3834651071497194(0)	0.6029496682837346(-1)
1.90	0.6870754132617678(0)	0.4593713686953585(1)	0.3757765396688848(0)	0.5620437817453485(-1)
1.95	0.6789003963937472(0)	0.4771778784589879(1)	0.3684042987138854(0)	0.5241437956799866(-1)
2.00	0.6704827097900733(0)	0.4954234356001890(1)	0.3613286168882226(0)	0.4890051070806112(-1)
2.05	0.6618658628802201(0)	0.5141308568493948(1)	0.3545313797517170(0)	0.4566056299118849(-1)
2.10	0.6530895931759691(0)	0.5333235359655902(1)	0.3479959534707185(0)	0.4261434150851507(-1)
2.15	0.6441870969978426(0)	0.5530255009015579(1)	0.3417070338905328(0)	0.3980345604801850(-1)
2.20	0.6351918144137759(0)	0.5732614699814381(1)	0.3356505134529960(0)	0.3719113705193257(-1)
2.25	0.6261313739651271(0)	0.5940569075037338(1)	0.3298133666814699(0)	0.3476207311944460(-1)
2.30	0.6170311019689313(0)	0.6154380791333508(1)	0.3241835429648885(0)	0.3250226717162159(-1)
2.35	0.6079136605259716(0)	0.6374321074059444(1)	0.3187498781096143(0)	0.3039890883162248(-1)
2.40	0.5987992978083817(0)	0.6600670276350539(1)	0.3135020126067469(0)	0.2844026093579567(-1)
2.45	0.5897060737337516(0)	0.6833718444858624(1)	0.3084303187941878(0)	0.2661555839389467(-1)
2.50	0.5806500637222667(0)	0.7073765894578601(1)	0.3035258364859841(0)	0.2491491787026973(-1)
2.55	0.5716455429153259(0)	0.7321237950133331(1)	0.2987802152032416(0)	0.2332925697125217(-1)
2.60	0.5627051529145893(0)	0.7576114769787598(1)	0.2941856623768447(0)	0.2185022180408219(-1)
2.65	0.5538400528724073(0)	0.7839073511713031(1)	0.2897348960868591(0)	0.2047012192534298(-1)
2.70	0.5450600565375012(0)	0.8110347415222037(1)	0.2854211067605018(0)	0.1918187182658638(-1)
2.75	0.5363737566762730(0)	0.8390297228026057(1)	0.2812379121729163(0)	0.1797893821545441(-1)
2.80	0.5277886381277471(0)	0.8679297723809165(1)	0.2771793304851186(0)	0.1685529244521605(-1)
2.85	0.5193111806092019(0)	0.8977738397739503(1)	0.2732397457133851(0)	0.1580536752680431(-1)
2.90	0.5109469522665601(0)	0.9286024186566349(1)	0.2694138803956638(0)	0.1482401922726118(-1)

2.95	0.5027006948559761(0)	0.9604576215068175(1)	0.2656967701046732(0)	0.1390649081872684(-1)	
3.00	0.4945764013486412(0)	0.9933832570625417(1)	0.2620837402553185(0)	0.1304838109419704(-1)	
3.05	0.4865773866677791(0)	0.1027424910770988(2)	0.2585703849668428(0)	0.1224561531143291(-1)	
3.10	0.4787063521935835(0)	0.1062630028410981(2)	0.2551525477685169(0)	0.1149441876565585(-1)	6.80
3.15	0.4709654446071541(0)	0.1099048003074495(2)	0.2518263039623042(0)	0.1079129272585048(-1)	6.90
3.20	0.4633563095871836(0)	0.1136730265696873(2)	0.2485879444773646(0)	0.1013299249934911(-1)	7.00
3.25	0.4558801408222987(0)	0.1175730379330493(2)	0.2454339610699440(0)	0.9516507415488778(-2)	7.10
3.30	0.4485377247567375(0)	0.1216104137362286(2)	0.2423610327385172(0)	0.8939042542032138(-2)	7.20
3.35	0.4413294814467530(0)	0.1257909665881873(2)	0.2393660132383432(0)	0.8398001968161591(-2)	7.30
3.40	0.4342555018691712(0)	0.1301207530414045(2)	0.2364459195921340(0)	0.7890973505565841(-2)	7.40
3.45	0.4273155819913683(0)	0.1346060847236905(2)	0.2335979215045578(0)	0.7415714674756149(-2)	7.50
3.50	0.4205092538831298(0)	0.1392535399515234(2)	0.2308193315980103(0)	0.6970139857548393(-2)	7.60
3.55	0.4138358141250072(0)	0.1440699758487413(2)	0.2281075963956555(0)	0.6552308508859890(-2)	7.70
3.60	0.4072943497445684(0)	0.1490625409953709(2)	0.2254602879853118(0)	0.6160414331797309(-2)	7.80
3.65	0.4008837618910396(0)	0.1542386886323746(2)	0.2228750963044691(0)	0.5792775329626956(-2)	7.90
3.70	0.3946027874400052(0)	0.1596061904491687(2)	0.2203498219926746(0)	0.5447824656770463(-2)	8.00
3.75	0.3884500187028375(0)	0.1651731509818928(2)	0.2178823697628117(0)	0.5124102198586810(-2)	8.10
3.80	0.3824239214001773(0)	0.1709480226516090(2)	0.2154707422475070(0)	0.4820246816497640(-2)	8.20
3.85	0.3765228510448905(0)	0.1769396214728705(2)	0.2131130342810929(0)	0.4534989201084495(-2)	8.30
3.90	0.3707450678673489(0)	0.1831571434644303(2)	0.2108074275813037(0)	0.4267145281218571(-2)	8.40
3.95	0.3650887504044733(0)	0.1896101817952653(2)	0.2085521857982297(0)	0.4015610142149997(-2)	8.50
4.00	0.3595520078636207(0)	0.1963087447005622(2)	0.2063456499010558(0)	0.3779352409848906(-2)	8.60
4.05	0.3541328913629877(0)	0.2032632742038675(2)	0.2041862338758001(0)	0.3557409062811431(-2)	8.70
4.10	0.3488294041416436(0)	0.2104846656832318(2)	0.2020724207096828(0)	0.3348808636069710(-2)	8.80
4.15	0.3436395108245115(0)	0.2179842883208902(2)	0.2000027586399285(0)	0.3152926785319663(-2)	8.90
4.20	0.3385611458205194(0)	0.2257740064778208(2)	0.1979758576467600(0)	0.2968762181941948(-2)	9.00
4.25	0.3335922209256595(0)	0.2338662020364075(2)	0.1959903861721022(0)	0.2795652712273958(-2)	9.10
4.30	0.3287306321967914(0)	0.2422737977564123(2)	0.1940456804711100(0)	0.2632911956823758(-2)	9.20
4.35	0.3239742661566163(0)	0.2510102816915399(2)	0.1921386796130670(0)	0.2479897927226724(-2)	9.30
4.40	0.3193210053853184(0)	0.2600897327160515(2)	0.1902700470215048(0)	0.2336010040655829(-2)	9.40
4.45	0.3147687335498526(0)	0.2695268472131682(2)	0.1884380437005717(0)	0.2200686313127430(-2)	9.50
4.50	0.3103153399177213(0)	0.2793369669793958(2)	0.1866415879757465(0)	0.2073400754714614(-2)	9.60
4.55	0.3059587233982997(0)	0.2895361084014081(2)	0.1848796408339623(0)	0.1953660951105750(-2)	9.70
4.60	0.3016967961512960(0)	0.3001409929647570(2)	0.1831512038210900(0)	0.1841005817241503(-2)	9.80
4.65	0.2975274867987521(0)	0.3111690791564253(2)	0.1814553170635290(0)	0.1735003509942023(-2)	9.90
4.70	0.2934487432740670(0)	0.3226385958261276(2)	0.1797910574053859(0)	0.1635249487508885(-2)	10.00
4.75	0.2894585353388472(0)	0.3345685770742831(2)	0.1781575366533868(0)	0.1541364705263895(-2)	10.20
4.80	0.2855548567959244(0)	0.3469788987377533(2)	0.1765538999222755(0)	0.1452999393687832(-2)	10.40
4.85	0.2817357274246173(0)	0.3598903165477514(2)	0.1749793240740063(0)	0.1369804212159651(-2)	10.60
4.90	0.2779991946622346(0)	0.3733245060378098(2)	0.1734330162445464(0)	0.1291483362705985(-2)	10.80
4.95	0.2743433350539052(0)	0.3873041042833270(2)	0.1719142124525677(0)	0.1217738667517481(-2)	11.00
5.00	0.2707662554910572(0)	0.4018527535580318(2)	0.1704221762847322(0)	0.1148295591275326(-2)	11.20
5.10	0.2638410218985874(0)	0.4327570763576145(2)	0.1675155916170776(0)	0.1021300107861738(-2)	11.40
5.20	0.2572089914235682(0)	0.4662485050579675(2)	0.1647078767047833(0)	0.9086216124486596(-3)	11.60
5.30	0.2508561971769137(0)	0.5025573030483023(2)	0.1619940141687699(0)	0.8086083340807593(-3)	11.80
5.40	0.2447692200166432(0)	0.5419347580097915(2)	0.1593693236120981(0)	0.7198044498634734(-3)	12.00
5.50	0.2389351937712944(0)	0.5846551424982394(2)	0.1568294335801275(0)	0.6409260498657626(-3)	12.20
5.60	0.2333418047996327(0)	0.6310178597429926(2)	0.1543702562925014(0)	0.5708401696482117(-3)	12.40
5.70	0.2279772869743245(0)	0.6813497923753799(2)	0.1519879648307450(0)	0.5085464802750316(-3)	12.60
5.80	0.2228304130069319(0)	0.7360078735067486(2)	0.1496789725060241(0)	0.4531612834935893(-3)	12.80
5.90	0.2178904828866256(0)	0.7953819014488757(2)	0.1474399141665211(0)	0.4039035089431294(-3)	13.00
6.00	0.2131473100815936(0)	0.8598976214243920(2)	0.1452676292338869(0)	0.3600824521626587(-3)	13.20
6.10	0.2085912060470571(0)	0.9300200998696368(2)	0.1431591462840558(0)	0.3210870279496547(-3)	13.40
6.20	0.2042129634943658(0)	0.1006257419406609(3)	0.1411116690100267(0)	0.2863763442277715(-3)	13.60
6.30	0.2000038387995428(0)	0.1089164725275854(3)	0.1391225634235193(0)	0.2554714278071156(-3)	13.80
6.40	0.1959555338649284(0)	0.1179348657001819(3)	0.1371893461691771(0)	0.2279479559293729(-3)	14.00
6.50	0.1920601776925695(0)	0.1277472202332260(3)	0.1353096738395544(0)	0.2034298668393982(-3)	14.20
6.60	0.1883103078813109(0)	0.1384260014082264(3)	0.1334813331918336(0)	0.1815837392834511(-3)	14.40
6.70	0.1846988522199469(0)	0.1500504234452441(3)	0.1317022321783076(0)	0.1621138451969451(-3)	14.60
					14.80
					15.00
					15.20

6.80	0.18121911105152683(0)	0.16270708757143143(3)	0.12997039171237769(0)	0.14475779224489963(-3)
6.90	0.17786473676550084(0)	0.17649068109015773(3)	0.12828393681003267(0)	0.12928268359847033(-3)
7.00	0.17462972176579023(0)	0.19150474333355014(3)	0.12664109607663283(0)	0.11548173161033823(-3)
7.10	0.17150837621207653(0)	0.20786250496872263(3)	0.12504018238713863(0)	0.10317172009369473(-3)
7.20	0.16849531435325933(0)	0.22568780770106383(3)	0.12347959987025343(0)	0.92188116887162053(-4)
7.30	0.16558543822692623(0)	0.24511611221637413(3)	0.12195783199146123(0)	0.82387244448808363(-4)
7.40	0.16277392250261223(0)	0.26629560282363723(3)	0.12047343779097653(0)	0.73639721492506183(-4)
7.50	0.16005619994661743(0)	0.28938839820014463(3)	0.11902504720841113(0)	0.65830893267080233(-4)
7.60	0.15742794751447163(0)	0.31457187849808363(3)	0.11761139567519013(0)	0.58858772075383903(-4)
7.70	0.15488507307057393(0)	0.34204014008870543(3)	0.11623112548232223(0)	0.52632613115921793(-4)
7.80	0.15242370272923343(0)	0.37200559031781583(3)	0.11488317128606673(0)	0.47071653772027773(-4)
7.90	0.15004016880707373(0)	0.40470069585337433(3)	0.11356663674123763(0)	0.42103997131542003(-4)
8.00	0.14773099837340103(0)	0.44037989953483833(3)	0.11227963925349933(0)	0.37665622843924903(-4)
8.10	0.14549290238251393(0)	0.47932172207958043(3)	0.11102196134795633(0)	0.33699510455383913(-4)
8.20	0.14332276536994743(0)	0.52183106662143303(3)	0.10979235465890153(0)	0.30154862143520373(-4)
8.30	0.14121763569318653(0)	0.56824174578666913(3)	0.10858988362210733(0)	0.26986413333787873(-4)
8.40	0.13917471629636913(0)	0.61891925296214423(3)	0.10741365644163723(0)	0.24153821048881293(-4)
8.50	0.13719135597785283(0)	0.67426380151833723(3)	0.10626281199684803(0)	0.21621121043483373(-4)
8.60	0.13526504113917563(0)	0.73471365807971103(3)	0.10513653832498263(0)	0.19356245831782773(-4)
8.70	0.13339338779938503(0)	0.80074879849012433(3)	0.10403405031625463(0)	0.17330596642261533(-4)
8.80	0.13157413521454403(0)	0.87289491792828453(3)	0.10295459798464073(0)	0.15518663149438383(-4)
8.90	0.12980513699746743(0)	0.95172782971194953(3)	0.10189746270431053(0)	0.13897685549481703(-4)
9.00	0.12808435652321393(0)	0.10378782907170903(4)	0.10086195558064093(0)	0.12447354178006273(-4)
9.10	0.12640985979379033(0)	0.11320372950595773(4)	0.99847415921481533(-1)	0.11149542424520993(-4)
9.20	0.12477980982617963(0)	0.12349618817752223(4)	0.98853209802604043(-1)	0.99880691881318103(-5)
9.30	0.12319246118342123(0)	0.13474815067254053(4)	0.97878728719977633(-1)	0.89484875512845683(-5)
9.40	0.12164615468488953(0)	0.14705050338901863(4)	0.96923388322986573(-1)	0.80178967296016673(-5)
9.50	0.12013931325817103(0)	0.16050284006323393(4)	0.95986627222929243(-1)	0.71847746923848303(-5)
9.60	0.11867043597568003(0)	0.17521436054727383(4)	0.95067905871612343(-1)	0.64388291455157313(-5)
9.70	0.11723809515989803(0)	0.19130451854627693(4)	0.94166705505261393(-1)	0.57708648310099093(-5)
9.80	0.11584043166186553(0)	0.20890458134314213(4)	0.93282527149333813(-1)	0.51726653295062713(-5)
9.90	0.11447765104829693(0)	0.22815819932875373(4)	0.92414890680192603(-1)	0.46368877571346823(-5)
10.00	0.11314702047941083(0)	0.24922289762418783(4)	0.91563333939788083(-1)	0.41569689296853243(-5)
10.20	0.11057906309321193(0)	0.29749291103181703(4)	0.89906695873394793(-1)	0.33418605374509443(-5)
10.40	0.10812029735387463(0)	0.35540553758632183(4)	0.88309244377819623(-1)	0.26874695790129603(-5)
10.60	0.10578674560512343(0)	0.42457338372030443(4)	0.86767849415708283(-1)	0.21619085804233913(-5)
10.80	0.10356713257697543(0)	0.50759633942984643(4)	0.85279598666984143(-1)	0.17396614639105173(-5)
11.00	0.10145028126361853(0)	0.60714063740986123(4)	0.83841778860345953(-1)	0.14003003042474423(-5)
11.20	0.09934770513270863(-1)	0.72653415846543443(4)	0.82451858998811713(-1)	0.11274628659297063(-5)
11.40	0.09737623811909753(-1)	0.86978134031899563(4)	0.81107475258314233(-1)	0.90803750987527693(-6)
11.60	0.09548329579931653(-1)	0.10417017143937323(5)	0.79806417367264443(-1)	0.73151262350394123(-6)
11.80	0.09366417524513863(-1)	0.12480971738111723(5)	0.78546616300124713(-1)	0.58945763105760633(-6)
12.00	0.09191454543889653(-1)	0.14959532666397533(5)	0.77326133138919233(-1)	0.47510818246724943(-6)
12.20	0.09023041162183963(-1)	0.17936826933584883(5)	0.76143148957517273(-1)	0.38303473070527973(-6)
12.40	0.08860808379858203(-1)	0.21514186069722723(5)	0.74995955740342533(-1)	0.30887749642302263(-6)
12.60	0.08704148071050583(-1)	0.25813687669555103(5)	0.73882947866423933(-1)	0.24913442574866933(-6)
12.80	0.08553544145078413(-1)	0.30982429453675253(5)	0.72802461469016543(-1)	0.20099146180920133(-6)
13.00	0.08407902916722513(-1)	0.37197688490688903(5)	0.71753533524462403(-1)	0.16218662188014333(-6)
13.20	0.08267218437250663(-1)	0.44673149720496503(5)	0.70734363329446523(-1)	0.13090109744877023(-6)
13.40	0.08131236940861613(-1)	0.53666426374343763(5)	0.69743838923514463(-1)	0.10567196690323013(-6)
13.60	0.07999722212377193(-1)	0.64488814140122463(5)	0.68780765680863203(-1)	0.85322201422766333(-7)
13.80	0.07872453585089003(-1)	0.77512895276063123(5)	0.67844014668983893(-1)	0.68904516416974673(-7)
14.00	0.07749225149209303(-1)	0.93192513633965373(5)	0.66932518183439633(-1)	0.55656311111145183(-7)
14.20	0.07629844291821436(-1)	0.11207205052893223(6)	0.66045265642834483(-1)	0.44963493635437853(-7)
14.40	0.07514130671741853(-1)	0.13480912276233863(6)	0.65181299810350353(-1)	0.36331428770754553(-7)
14.60	0.07401915244119631(-1)	0.16219727130720857(6)	0.64339713311897843(-1)	0.29361597820222933(-7)
14.80	0.07293039376885267(-1)	0.19519419117756394(6)	0.63519645424016363(-1)	0.23732841180828413(-7)
15.00	0.07187354049241071(-1)	0.23495585249076833(6)	0.62720279107409243(-1)	0.19186278921478673(-7)
15.20	0.07084719123528884(-1)	0.28287773247593373(6)	0.61940838264435683(-1)	0.15513184354897633(-7)

15.40	0.6985002682813623(-1)	0.3406446704365455(6)	0.6118058520104437(-1)	0.1254522935050609(- 7)	38.00
15.60	0.6888080427400526(-1)	0.4102911290368417(6)	0.6043881827555632(-1)	0.1014663520014089(- 7)	38.50
15.80	0.6793835124275987(-1)	0.4942740579976479(6)	0.5971486971841596(-1)	0.8207854989978216(- 8)	39.00
16.00	0.6702156104139909(-1)	0.5955609986708370(6)	0.5900810360855644(-1)	0.6640487249441043(- 8)	39.50
16.20	0.6612938801293519(-1)	0.7177366287113441(6)	0.5831791399338751(-1)	0.5373179810396308(- 8)	40.00
16.40	0.6526084332170970(-1)	0.8651316186170387(6)	0.5764372314063416(-1)	0.4348330246413268(- 8)	40.50
16.60	0.6441499108764715(-1)	0.1042978487220513(7)	0.5698497991134497(-1)	0.3519427311392819(- 8)	41.00
16.80	0.6359094483601590(-1)	0.1257600130511370(7)	0.5634115024436879(-1)	0.2848908328641833(- 8)	41.50
17.00	0.6278786423285517(-1)	0.1516637894042517(7)	0.5571175574347673(-1)	0.2306431989821655(- 8)	42.00
17.20	0.6200495207939739(-1)	0.1829327507651212(7)	0.5509629235909709(-1)	0.1867485703456464(- 8)	42.50
17.40	0.6124145154161804(-1)	0.2206832955834809(7)	0.5449430915734226(-1)	0.1512262441400273(- 8)	43.00
17.60	0.6049664359352493(-1)	0.2662650482689691(7)	0.5390536716984754(-1)	0.1224754734668012(- 8)	43.50
17.80	0.5976984465499713(-1)	0.3213097505446692(7)	0.5332904631692088(-1)	0.9920236807498873(- 9)	44.00
18.00	0.5906040440693245(-1)	0.3877904330597444(7)	0.5276494440262507(-1)	0.8036090344828678(- 9)	44.50
18.20	0.5836770376819408(-1)	0.4680903446916436(7)	0.5221267616968677(-1)	0.6510530577803806(- 9)	45.00
18.40	0.5769115302038587(-1)	0.5651030950391941(7)	0.5167187241655524(-1)	0.5275161160153598(- 9)	45.50
18.60	0.5703019006785616(-1)	0.6823107006991331(7)	0.5114217916812135(-1)	0.4274662841215528(- 9)	46.00
18.80	0.5638427882155116(-1)	0.8239375386198908(7)	0.5062325689756015(-1)	0.3464286669773704(- 9)	46.50
19.00	0.5575290769642903(-1)	0.9950907251046845(7)	0.5011477979547992(-1)	0.2807829097060795(- 9)	47.00
19.20	0.5513558821311919(-1)	0.1201949067167750(8)	0.4961643508305196(-1)	0.2275996053466976(- 9)	47.50
19.40	0.5453185369538304(-1)	0.1451988610649781(8)	0.4912792236605991(-1)	0.1845081053098080(- 9)	48.00
19.60	0.5394125805571199(-1)	0.1754255781301086(8)	0.4864895302704905(-1)	0.1495896868258273(- 9)	48.50
19.80	0.5336337466209903(-1)	0.2119698184951045(8)	0.4817924965297609(-1)	0.1212911993048645(- 9)	49.00
20.00	0.5279779527964813(-1)	0.2561565266405659(8)	0.4771854549596084(-1)	0.9835525290649882(-10)	49.50
20.50	0.5143515241690600(-1)	0.4114309041706761(8)	0.4660449650238127(-1)	0.5826274488896287(-10)	50.00
21.00	0.5014133864682561(-1)	0.6612718635548492(8)	0.4554136165450517(-1)	0.3453201267146756(-10)	51.00
21.50	0.4891124541721118(-1)	0.1063506337502374(9)	0.4452572364680038(-1)	0.2047762688262039(-10)	52.00
22.00	0.4774025998569045(-1)	0.1711446713003637(9)	0.4355444669261788(-1)	0.1214937895620437(-10)	53.00
22.50	0.4662420634887792(-1)	0.2755734002959935(9)	0.4262473437505852(-1)	0.7211669954118324(-11)	54.00
23.00	0.4555929447721889(-1)	0.4439663698302712(9)	0.4173392155600062(-1)	0.4282684795665672(-11)	55.00
23.50	0.4454207651365593(-1)	0.7156337008899201(9)	0.4087962975740886(-1)	0.2544407277410374(-11)	56.00
24.00	0.4356940883854058(-1)	0.1154115391849183(10)	0.4005965552384032(-1)	0.1512305893999706(-11)	57.00
24.50	0.4263841909814872(-1)	0.1862155313713956(10)	0.3927196935293355(-1)	0.8992239668453620(-12)	58.00
25.00	0.4174647745066453(-1)	0.3005959096525549(10)	0.3851469884490402(-1)	0.5348899755340217(-12)	59.00
25.50	0.4089117140987806(-1)	0.4854436678520199(10)	0.3778611377527881(-1)	0.3182899598082895(-12)	60.00
26.00	0.4007028376965511(-1)	0.7842940991898187(10)	0.3708461283934969(-1)	0.1894685885674978(-12)	61.00
26.50	0.3928177317438785(-1)	0.1267636845388481(11)	0.3640871185375504(-1)	0.1128240131086243(-12)	62.00
27.00	0.3852375697492631(-1)	0.2049649711988081(11)	0.3575703323153753(-1)	0.6720637435262040(-13)	63.00
27.50	0.3779449605489522(-1)	0.3315330364332410(11)	0.3512829657302925(-1)	0.4004597190457930(-13)	64.00
28.00	0.3709238137315206(-1)	0.5364511859231470(11)	0.3452131023684736(-1)	0.2386941511933733(-13)	65.00
28.50	0.3641592199588499(-1)	0.8683284776434690(11)	0.3393496377383885(-1)	0.1423163037419925(-13)	66.00
29.00	0.3576373442994828(-1)	0.1405991957584069(12)	0.3336822112256107(-1)	0.8487759778353563(-14)	67.00
29.50	0.3513453309465803(-1)	0.2277306066537837(12)	0.3282011447828864(-1)	0.5063524030714399(-14)	68.00
30.00	0.3452712179236185(-1)	0.3689732094072742(12)	0.3228973875898013(-1)	0.3021552010688812(-14)	69.00
30.50	0.3394038606740911(-1)	0.5979962712214315(12)	0.3177624660143239(-1)	0.1803519735842456(-14)	70.00
31.00	0.3337328627949772(-1)	0.9694555759683940(12)	0.3127884382925677(-1)	0.1076767038616238(-14)	71.00
31.50	0.3282485151125753(-1)	0.1572095592474019(13)	0.3079678534154509(-1)	0.6430269957499400(-15)	72.00
32.00	0.3229417388175753(-1)	0.2550043566357787(13)	0.3032937137733325(-1)	0.3840961801225067(-15)	73.00
32.50	0.3178040354772841(-1)	0.4137424382005289(13)	0.2987594411636582(-1)	0.2294832419214651(-15)	74.00
33.00	0.3128274412294852(-1)	0.6714640184076497(13)	0.2943588458134151(-1)	0.1371384348448747(-15)	75.00
33.50	0.3080044853367856(-1)	0.1089989174729090(14)	0.2900860981088176(-1)	0.8197129056219255(-16)	76.00
34.00	0.3033281525443544(-1)	0.1769803724411627(14)	0.2859357027600140(-1)	0.4900676118392788(-16)	77.00
34.50	0.2987918488392727(-1)	0.2874275355830471(14)	0.2819024751594533(-1)	0.2930483385121722(-16)	78.00
35.00	0.2943893702575893(-1)	0.4669055014466160(14)	0.2779815197195296(-1)	0.1752705938994737(-16)	79.00
35.50	0.2901148744267182(-1)	0.7586196776515777(14)	0.2741682099987429(-1)	0.1048486852619120(-16)	80.00
36.00	0.2859628545669380(-1)	0.1232852079912098(15)	0.2704581704463538(-1)	0.6273339009762242(-17)	
36.50	0.2819281157072319(-1)	0.2003950446679199(15)	0.2668472596137432(-1)	0.3754171926217522(-17)	
37.00	0.2780057528982065(-1)	0.3257988998672264(15)	0.2633315546967471(-1)	0.2247020697588571(-17)	
37.50	0.2741911312288861(-1)	0.5297811170425472(15)	0.2599073372874117(-1)	0.1345164720542498(-17)	

38.00	0.2704798674752830(-1)	0.8616388199965787(15)	0.2565710802261442(-1)	0.8054106916290750(-18)
38.50	0.2668678132271802(-1)	0.1401631180584185(16)	0.2533194354563262(-1)	0.4823152104821002(-18)
39.00	0.2633510393558843(-1)	0.2280446200301903(16)	0.2501492227933002(-1)	0.2888779301522701(-18)
39.50	0.2599258217000951(-1)	0.3710918879133971(16)	0.2470574195283762(-1)	0.1730477137053579(-18)
40.00	0.2565886278597515(-1)	0.6039718263611241(16)	0.2440411507962858(-1)	0.1036773261451657(-18)
40.50	0.2533361049989644(-1)	9.9831586535606510(16)	0.2410976806414384(-1)	0.6212501625936037(-19)
41.00	0.2501650685691183(-1)	1.6100664914324504(17)	0.2382244037245185(-1)	0.3723166776459978(-19)
41.50	0.2470724918720716(-1)	2.6906425970774542(17)	0.2354188376164872(-1)	0.2231619830976716(-19)
42.00	0.2440554963912712(-1)	4.4244796092136851(17)	0.2326786156319983(-1)	0.1337790881001177(-19)
42.50	0.2411113428255924(-1)	6.6914059660093667(17)	0.2300014801586681(-1)	0.8020753140591072(-20)
43.00	0.2382374227669796(-1)	0.1126348290166967(18)	0.2273852764426145(-1)	0.4809496556950018(-20)
43.50	0.2354312509685454(-1)	0.1835160585140183(18)	0.2248279467942528(-1)	0.2884299346610979(-20)
44.00	0.2326904581547836(-1)	0.2990444718632337(18)	0.2223275251815504(-1)	0.1729959874281648(-20)
44.50	0.2300127843300251(-1)	0.4873673371763902(18)	0.2198821321808334(-1)	0.1037732683120525(-20)
45.00	0.2273960725452828(-1)	0.7943916035704456(18)	0.2174899702578527(-1)	0.6225690809462383(-21)
45.50	0.2248382630872347(-1)	0.1294998150712611(19)	0.2151493193541692(-1)	0.3735433849181410(-21)
46.00	0.2223373880563390(-1)	0.2111342388647824(19)	0.2128585327560503(-1)	0.2241531759744300(-21)
46.50	0.2198915663039922(-1)	0.3442722209946245(19)	0.2106160332249963(-1)	0.1345234573407572(-21)
47.00	0.2174989987012736(-1)	0.5614329680810343(19)	0.2084203093707619(-1)	0.8074197842725814(-22)
47.50	0.2151579637141969(-1)	0.9156833466011563(19)	0.2062699122493194(-1)	0.4846720698787642(-22)
48.00	0.2128668132625326(-1)	0.1493630213112993(20)	0.2041634521696516(-1)	0.2909664190405842(-22)
48.50	0.2106239688412107(-1)	0.2436633239310381(20)	0.2020995595695640(-1)	0.1746960447705315(-22)
49.00	0.2084279178850680(-1)	0.3975442747903745(20)	0.2000770628218984(-1)	0.1048981164236802(-22)
49.50	0.2062772103593020(-1)	0.6486764087887549(20)	0.1980946243336086(-1)	0.6299351408664804(-23)
50.00	0.2041704555594399(-1)	0.1058536389713169(21)	0.1961510993011487(-1)	0.3783264029550459(-23)
51.00	0.2000835201197715(-1)	0.2819875254895520(21)	0.1923762933791553(-1)	0.1365001019107898(-23)
52.00	0.1961570627560364(-1)	0.7514792768154392(21)	0.1887440774812071(-1)	0.4926747149729344(-24)
53.00	0.1923818093493668(-1)	0.2003417881855170(22)	0.1852465196867329(-1)	0.1778863059762939(-24)
54.00	0.1887491871954997(-1)	0.5343023851440743(22)	0.1818762658359069(-1)	0.6425012932971142(-25)
55.00	0.1852512598949623(-1)	0.1425468664988751(23)	0.1786266878519243(-1)	0.2321396652668892(-25)
56.00	0.1818806993669695(-1)	0.3804324292226218(23)	0.1754908375138907(-1)	0.8390028962572610(-26)
57.00	0.1786305840936042(-1)	0.1015643484474065(24)	0.1724634050210516(-1)	0.3033272919545714(-26)
58.00	0.1754946528267050(-1)	0.2712338189315405(24)	0.1695386817786948(-1)	0.1096955100924602(-26)
59.00	0.1724669630961408(-1)	0.7245699996696615(24)	0.1667115269121852(-1)	0.3968178474512270(-27)
60.00	0.1695420039481329(-1)	0.1936182213929277(25)	0.1639771370804653(-1)	0.1435867565681257(-27)
61.00	0.1667146324187138(-1)	0.5175318892687175(25)	0.1613310192157824(-1)	0.5197021115800324(-28)
62.00	0.1639800433125279(-1)	0.1383722092026783(26)	0.1587689658639005(-1)	0.1881515229760728(-28)
63.00	0.1613337419128045(-1)	0.3700646202538074(26)	0.1562870328398534(-1)	0.6813505112488122(-29)
64.00	0.1587715192959883(-1)	0.9899640925974456(26)	0.1538815189494344(-1)	0.2467968559452694(-29)
65.00	0.1562894299654450(-1)	0.2648932738634548(27)	0.1515489475569561(-1)	0.8941525120697856(-30)
66.00	0.1538837715539018(-1)	0.7089712555436261(27)	0.1492860498060671(-1)	0.3240286570844400(-30)
67.00	0.1515510663747108(-1)	0.1897969740295516(28)	0.1470897493231828(-1)	0.1174497563867716(-30)
68.00	0.1492880446283658(-1)	0.5082177148694186(28)	0.1449571482528862(-1)	0.4258090299574639(-31)
69.00	0.1470916290933767(-1)	0.1361153829291226(29)	0.1428855144918970(-1)	0.1544076991753565(-31)
70.00	0.1449589211510103(-1)	0.3644352759579736(29)	0.1408722700032681(-1)	0.5600306285813438(-32)
71.00	0.1428871880095736(-1)	0.9770156145547613(29)	0.1389149801056327(-1)	0.2031612451633665(-32)
72.00	0.1408738510104050(-1)	0.2618382465224057(30)	0.1370113436438754(-1)	0.7371465348386786(-33)
73.00	0.1389164749098190(-1)	0.7018607129747796(30)	0.1351591835577345(-1)	0.2675151499435887(-33)
74.00	0.1370127580433766(-1)	0.1881709898766298(31)	0.1333564405737618(-1)	0.9710069409656658(-34)
75.00	0.1351605232888535(-1)	0.5045869418512775(31)	0.1316011615539307(-1)	0.3525117355552349(-34)
76.00	0.1333577097532315(-1)	0.1353314556155587(32)	0.1298914964411187(-1)	0.1279970898341254(-34)
77.00	0.1316023651169158(-1)	0.3630268948220829(32)	0.1282256897478357(-1)	0.4648361947902859(-35)
78.00	0.1298926385753334(-1)	0.9739891688771744(32)	0.1266020749400019(-1)	0.1688384028133141(-35)
79.00	0.1282267743242176(-1)	0.2613622063250456(33)	0.1250190688724092(-1)	0.6133553950256061(-36)
80.00	0.1266031055403288(-1)	0.7014600004904800(33)	0.1234751666367861(-1)	0.2228543258688473(-36)

In using this table each tabulated entry is to be multiplied by the power of 10 given in the parentheses located just to the right of the last significant digit listed.

The authors wish to thank Professor R. A. Matsen for his help and encouragement.

Esso Research and Engineering
Planning Engineering Division
Linden, New Jersey

The University of Wisconsin
Naval Research Laboratory
Madison, Wisconsin

1. See FRANK E. HARRIS, "Tables of the exponential integral Ei(x)," *MTAC*, v. 11, 1957, p. 9–16 for a summary of these methods. The authors are grateful to Dr. Harris for the use of his material prior to publication.

2. K. RUEDENBERG, C. C. J. ROOTHAAN, & W. JAUNZEMIS, "Study of two-center integrals useful in calculations of molecular structure. III. A unified treatment of the hybrid, coulomb, and one-electron integrals," *Jn. Chem. Phys.*, v. 24, 1956, p. 201–20.

3. M. KOTANI, A. AMEMIYA, E. ISHIGURO, & T. KIMURA, *Table of Molecular Integrals*, Maruzen Co., Ltd., Tokyo, 1955.

A Note on Summation Formulas of Powers of Roots

By Daniel C. Fielder

Introduction. Through application of a method due to Newton [1], it is possible to find solutions for the sum S_b of the bth powers of the n roots of $f(x)$, where $f(x)$ is an integral rational function of nth degree in x. As anyone who has used this method for large n and b can attest, the procedure involves tedious and repetitive manipulations which increase rapidly in complexity as b is increased.

It is shown herein that it is possible to reduce the labor of finding such algebraic sums by rejecting redundant information in a general formula for the sums and using only pertinent terms to predict the numerical and literal coefficients. Inductive reasoning is employed to arrive at a simplified method.

Conventional means for determining sums. Given the expression

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots a_{n-1}x + a_n, \tag{1}$$

where $f(x)$ is an integral rational function of nth degree in x, and a_0, a_1, etc., are the coefficients of the powers of x. The sum of the bth powers of the roots of $f(x)$ are

$$S_b = c_1^b + c_2^b + \cdots c_n^b, \tag{2}$$

where c_1, c_2, etc., are the roots of $f(x)$. Newton's method makes use of the following tabulation to find S_b.

$$\begin{aligned} &1a_1 + S_1a_0 = 0 \\ &2a_2 + S_1a_1 + S_2a_0 = 0 \\ &3a_3 + S_1a_2 + S_2a_1 + S_3a_0 = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{3}$$

For the case of $n = b$, to get S_n it is necessary to go through n operations, each successive one becoming increasingly involved. The only respite occurs if one is able to find a table of S_n's of desired scope. Unfortunately, available tables seem to end with the fourth or fifth summation. A typical listing of sums and a summary of Newton's method is given by Adams [1].

Development of a summation formula. A general summation formula is now presented. While in the interests of simplicity it is not desirable to use this formula directly to calculate S_n, it is found that a limited number of terms of this formula for a particular S_n can be exploited advantageously to yield complete information

Received 26 November 1957.

about the S_n. By induction, it can be shown

$$
\begin{aligned}
S_n = &-\frac{1}{a_0}(na_n) + \frac{1}{a_0^2}\left[\frac{n}{2}\sum_{A=1}^{n-1}(a_A a_{n-A})\right] \\
&-\frac{1}{a_0^3}\left[\frac{n}{3}\sum_{B=1}^{n-2}\sum_{A=1}^{n-1-B}(a_B a_A a_{n-B-A})\right] \\
&+\frac{1}{a_0^4}\left[\frac{n}{4}\sum_{C=1}^{n-3}\sum_{B=1}^{n-2-C}\sum_{A=1}^{n-1-C-B}(a_C a_B a_A a_{n-C-B-A})\right] \qquad (4) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&+\frac{(-1)^n}{a_0^n}\Bigg[\frac{n}{n}\underbrace{\sum_{Z=1}^{1}\sum_{Y=1}^{2-Z}\sum_{X=1}^{3-Z-Y}}_{(n-1)\sum\text{'s}}\cdots\underbrace{(a_Z a_Y a_X\cdots a_A a_{n-Z-Y-X\cdots-A})}_{n\text{ literal terms}}\Bigg].
\end{aligned}
$$

One needs only to consider S_n for n equal to 11 or 12, for example, to note that the use of equation (4) results in summations of many repetitive combinations of a's. However, it can be seen that S_n is an nth order integral symmetric function [2] in a_1, a_2, etc., with the weight of the symmetric function necessarily equal to n. For example, if n were nine, one of the literal terms of the coefficients of $-1/a_0^5$ in equation (4) would be $a_1^2a_2^2a_3^1$. The weight is the sum of the products of corresponding subscripts and exponents. In this case, it is $1 \times 2 + 2 \times 2 + 3 \times 1 = 9$. A second necessary condition is that the degree of the product of the a's (exclusive of a_0) must equal the power of the corresponding a_0. For example, $a_2^3a_3^1$ would not be found in the coefficient of any $1/a_0^5$. Thus, the coefficient of $(-1)^k/a_0^k$ contains all the combinations of a's (exclusive of a_0) which simultaneously have weight of n and degree of k.

Development of a practical summation form. To specify completely the coefficient for each $(-1)^k/a_0^k$ of a particular S_n, one must know (a) what the combinations of the a's are and (b) what the individual numerical coefficients of the combinations are. Obviously, equation (4), after required manipulations, yields the answer. However, if the tedious approach of equation (4) is to be avoided, the information must be obtained in a somewhat different manner from direct application of equation (4).

The answer to (a) above can be found from a slight rearrangement of a table which Euler first developed in connection with enumeration of partitions. The table is presented in extended form in Euler's "Introductio," Lib. 1, Chap. XVI, and is reproduced in Chrystal [3]. The number of literal coefficients for each $(-1)^k/a_0^k$ is found directly from the rearranged table which is shown herein as Table 1. A brief explanation of the mechanics of constructing the table is given at the end of this paper. The literal coefficients are found by systematically setting down combinations of the a's with weight and degree consistent with n and k until the number specified by Table 1 is formed. For example, if $n = 11$ and $k = 3$, Table 1 indicates that there are ten literal coefficients.

TABLE 1. *Enumeration of Coefficients of* $\frac{(-1)^k}{a_0{}^k}$ *for* S_n

Values of n	n	$n-1$	$n-2$	$n-3$	$n-4$	$n-5$	$n-6$	$n-7$	$n-8$	$n-9$	$n-10$	$n-11$	$n-12$	$n-13$	$n-14$	$n-15$	$n-16$
		Values of k															
1	1																
2	1	1															
3	1	1	1														
4	1	1	2	1													
5	1	1	2	2	1												
6	1	1	2	3	3	1											
7	1	1	2	3	4	3	1										
8	1	1	2	3	5	5	4	1									
9	1	1	2	3	5	6	7	4	1								
10	1	1	2	3	5	7	9	8	5	1							
11	1	1	2	3	5	7	10	11	10	5	1						
12	1	1	2	3	5	7	11	13	15	12	6	1					
13	1	1	2	3	5	7	11	14	18	18	14	6	1				
14	1	1	2	3	5	7	11	15	20	23	23	16	7	1			
15	1	1	2	3	5	7	11	15	21	26	30	27	19	7	1		
16	1	1	2	3	5	7	11	15	22	28	35	37	34	21	8	1	
17	1	1	2	3	5	7	11	15	22	29	38	44	47	39	24	8	1

The answer to (b) above can be found from a consideration of the exponents of the a's. In the coefficient of each $(-1)^k/a_0^k$, there always appears the combination $a_1^{k-1}a_{n-k+1}$ which has the numerical coefficient n. The product $n \times (k-1)! \times 1!$ divided by the product of the factorials of the exponents of a particular

TABLE 2. *Values of* S_n, $n = 1(1)11$

$$S_1 = -a_1/a_0,$$

$$S_2 = a_1^2/a_0^2 - 2a_2/a_0,$$

$$S_3 = -a_1^3/a_0^3 + 3a_1a_2/a_0^2 - 3a_3/a_0,$$

$$S_4 = a_1^4/a_0^4 - 4a_1^2a_2/a_0^3 + (4a_1a_3 + 2a_2^2)/a_0^2 - 4a_4/a_0,$$

$$S_5 = -a_1^5/a_0^5 + 5a_1^3a_2/a_0^4 - (5a_1^2a_3 + 5a_1a_2^2)/a_0^3 + (5a_1a_4 + 5a_2a_3)/a_0^2 - 5a_5/a_0,$$

$$S_6 = a_1^6/a_0^6 - 6a_1^4a_2/a_0^5 + (6a_1^3a_3 + 9a_1^2a_2^2)/a_0^4 - (6a_1^2a_4 + 12a_1a_2a_3 + 2a_2^3)/a_0^3 + (6a_1a_5 + 6a_2a_4 + 3a_3^2)/a_0^2 - 6a_6/a_0,$$

$$S_7 = -a_1^7/a_0^7 + 7a_1^5a_2/a_0^6 - (7a_1^4a_3 + 14a_1^3a_2^2)/a_0^5 + (7a_1^3a_4 + 21a_1^2a_2a_3 + 7a_1a_2^3)/a_0^4 - (7a_1^2a_5 + 14a_1a_2a_4 + 7a_2^2a_3 + 7a_1a_3^2)/a_0^3 + (7a_1a_6 + 7a_2a_5 + 7a_3a_4)/a_0^2 - 7a_7/a_0,$$

$$S_8 = a_1^8/a_0^8 - 8a_1^6a_2/a_0^7 + (8a_1^5a_3 + 20a_1^4a_2^2)/a_0^6 - (8a_1^4a_4 + 32a_1^3a_2a_3 + 16a_1^2a_2^3)/a_0^5 + (8a_1^3a_5 + 24a_1^2a_2a_4 + 12a_1^2a_3^2 + 24a_1a_2^2a_3 + 2a_2^4)/a_0^4 - (8a_1^2a_6 + 16a_1a_2a_5 + 16a_1a_3a_4 + 8a_2^2a_4 + 8a_2a_3^2)/a_0^3 + (8a_1a_7 + 8a_2a_6 + 8a_3a_5 + 4a_4^2)/a_0^2 - 8a_8/a_0,$$

$$S_9 = -a_1^9/a_0^9 + 9a_1^7a_2/a_0^8 - (9a_1^6a_3 + 27a_1^5a_2^2)/a_0^7 + (9a_1^5a_4 + 45a_1^4a_2a_3 + 30a_1^3a_2^3)/a_0^6 - (9a_1^4a_5 + 36a_1^3a_2a_4 + 18a_1^3a_3^2 + 54a_1^2a_2^2a_3 + 9a_1a_2^4)/a_0^5 + (9a_1^3a_6 + 27a_1^2a_2a_5 + 27a_1^2a_3a_4 + 27a_1a_2a_3^2 + 9a_2^3a_3)/a_0^4 - (9a_1^2a_7 + 18a_1a_2a_6 + 18a_1a_3a_5 + 9a_1a_4^2 + 9a_2^2a_5 + 18a_2a_3a_4 + 3a_3^3)/a_0^3 + (9a_1a_8 + 9a_2a_7 + 9a_3a_6 + 9a_4a_5)/a_0^2 - 9a_9/a_0,$$

$$S_{10} = a_1^{10}/a_0^{10} - 10a_1^8a_2/a_0^9 + (10a_1^7a_3 + 35a_1^6a_2^2)/a_0^8 - (10a_1^6a_4 + 60a_1^5a_2a_3 + 50a_1^4a_2^3)/a_0^7 + (10a_1^5a_5 + 50a_1^4a_2a_4 + 25a_1^4a_3^2 + 100a_1^3a_2^2a_3 + 25a_1^2a_2^4)/a_0^6 - (10a_1^4a_6 + 40a_1^3a_2a_5 + 40a_1^3a_3a_4 + 60a_1^3a_2^2a_4 + 60a_1^2a_2^3a_2 + 40a_2^3a_3a_1 + 2a_2^5)/a_0^5 + (10a_1^3a_7 + 30a_1^2a_2a_6 + 30a_1^2a_3a_5 + 15a_1^2a_4^2 + 30a_1a_2^2a_5 + 60a_1a_2a_3a_4 + 10a_2^3a_4 + 15a_2^2a_3^2 + 10a_1a_3^3)/a_0^4 - (10a_1^2a_8 + 20a_1a_3a_7 + 20a_1a_3a_6 + 20a_1a_4a_5 + 20a_2a_3a_5 + 10a_2a_4^2 + 10a_2^2a_6 + 10a_3^2a_4)/a_0^3 + (10a_1a_9 + 10a_2a_8 + 10a_3a_7 + 10a_4a_6 + 5a_5^2)/a_0^2 - 10a_{10}/a_0,$$

$$S_{11} = -a_1^{11}/a_0^{11} + 11a_1^9a_2/a_0^{10} - (11a_1^8a_3 + 44a_1^7a_2^2)/a_0^9 + (11a_1^7a_4 + 77a_1^6a_2a_3 + 77a_1^5a_2^3)/a_0^8 - (11a_1^6a_5 + 66a_1^5a_2a_4 + 33a_1^5a_3^2 + 165a_1^4a_2^2a_3 + 55a_1^3a_2^4)/a_0^7 + (11a_1^5a_6 + 55a_1^4a_2a_5 + 55a_1^4a_3a_4 + 110a_1^3a_2^2a_3 + 110a_1^3a_2^3a_4 + 110a_1^2a_2a_3^2 + 11a_1a_2^5)/a_0^6 - (11a_1^4a_7 + 44a_1^3a_2a_6 + 44a_1^3a_3a_5 + 22a_1^3a_4^2 + 66a_1^2a_2^2a_5 + 132a_1^2a_2a_3a_4 + 44a_1a_2^3a_4 + 66a_1a_2^2a_3^2 + 11a_2^4a_3 + 22a_1^2a_3^3)/a_0^5 + (11a_1^3a_8 + 33a_1^2a_2a_7 + 33a_1^2a_3a_6 + 33a_1^2a_4a_5 + 33a_1a_2^2a_6 + 66a_1a_2a_3a_5 + 33a_1a_2a_4^2 + 33a_1a_2^2a_4 + 11a_2^3a_5 + 33a_2^2a_3a_4 + 11a_3^3a_2)/a_0^4 - (11a_1^2a_9 + 22a_1a_2a_8 + 22a_1a_3a_7 + 22a_1a_4a_6 + 11a_1a_5^2 + 11a_2^2a_7 + 22a_2a_3a_6 + 22a_2a_4a_5 + 11a_3a_4^2 + 11a_3^2a_5)/a_0^3 + (11a_1a_{10} + 11a_2a_9 + 11a_3a_8 + 11a_4a_7 + 11a_5a_6)/a_0^2 - 11a_{11}/a_0.$$

combination of a's yields the numerical coefficient for that combination. For example, in the summation for S_{11} given in Table 2, $1/a_0^4$ has as the $a_1^{k-1}a_{n-k+1}$ coefficient $a_1^3a_8$ which, in turn, has the numerical coefficient 11. The $a_1^3a_2a_6$ combination has $2! \times 1! \times 1!$ as the product of the factorials of the exponents. If $11 \times 3! \times 1!$ is divided by $2! \times 1! \times 1!$, the result is 33 which is the numerical coefficient of the $a_1^3a_2a_6$ combination.

Construction of Table 1. Table 1 shows the number of coefficients of $-1/a_0$ as the extreme right-hand entry corresponding to a given n. The next entry to the left is the number of coefficients for $+1/a_0^2$, etc. The pattern for the number of $-1/a_0$ and $+1/a_0^2$ coefficients is obvious. It should also be noted that entries to the left of the stepped division remain fixed for n and k. The entries in a column for $k = (n - h)$, where $h = 0, 1, 2$, etc., are formed from cumulative sums, starting from the right, of the entries in the row for $n = h$. For example, for $n = 6$, $h = 5$, and $k = 1$, the entry is 1. This entry is equal to the first 1 on the right in the row for $n = 5$. For $n = 7$, $h = 5$, and $k = 2$, the entry is 3. This entry is equal to $(1 + 2)$ which is the sum of the first two right-hand entries of the row for $n = 5$. A column is formed in this fashion until the sum of the entries in the row (in this case, the fifth) is arrived at, after which the entries in the column remain fixed at this value. The fixed values are those to the left of the stepped division, as explained above.

Summary. The method of finding S_n may be summarized briefly as follows. First, obtain the number of coefficients of each $(-1)^k/a_0^k$ from Table 1 or an extension thereof. Second, by some systematic procedure obtain as many different combinations of the a's as there are coefficients specified by Table 1, making certain that the weight of each combination equals n and the degree equals k. Third, find $n \times (k - 1)!$ divided by the product of the factorials of the exponents to obtain the numerical value of a combination. A listing of S_n from $n = 1$ to $n = 11$ is given in Table 2.

Georgia Institute of Technology
Atlanta, Georgia

1. E. P. ADAMS, *Smithsonian Mathematical Formulae and Tables of Elliptic Functions*, Smithsonian Institution, Washington, 1947, p. 2–3.
2. G. CHRYSTAL, *Textbook of Algebra*, Vol. 1, Chelsea Publishing Co., New York, 1952, p. 434.
3. G. CHRYSTAL, *Textbook of Algebra*, Vol. 2, Chelsea Publishing Co., New York, 1952, p. 558.

An Alternating Direction Method for Solving The Biharmonic Equation

By S. D. Conte and R. T. Dames

1. Introduction. The biharmonic operator, defined by

$$\Delta\Delta = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4},$$

plays a role in the classical theory of plates similar to that of the Laplace operator in potential theory. In particular, the biharmonic equation, $\Delta\Delta W = 0$, together

Received 4 October 1957.

with appropriate boundary conditions, describes the small deflections in a thin homogeneous plate under various forces exerted on the plate boundary. From a computational point of view, direct numerical solutions of such boundary value problems are of considerable interest even in those cases where analytic solutions in series form are available.

The use of relaxation methods for the solution of the biharmonic equation has been considered by Frankel [3] and more recently by Held, Heller and Lubell [2]. It is the purpose of this paper to present an implicit alternating direction iterative scheme which appears to be more efficient than any of the known relaxation methods. The procedure is an extension of a method developed by Douglas and Rachford [1] for Poisson's equation.

2. The differential and difference equations. For definiteness we shall consider the problem of determining the deflections W in a square homogeneous plate supported along its boundaries and bent by moments distributed along two opposite edges. The boundary value problem associated with W may be written:

$$
\begin{aligned}
&\text{(a)} \quad \Delta\Delta W = \frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} = 0 \quad (x, y) \in R,\\
&\text{(b)} \quad W(x, y) = \frac{\partial^2 W}{\partial x^2} = 0 \quad \text{at } x = 0, 1 \quad \text{for } 0 < y < 1,\\
(2)\quad &\text{(c)} \quad W(x, y) = 0 \quad \text{at } y = 0, 1 \quad \text{for } 0 < x < 1,\\
&\text{(d)} \quad \frac{\partial^2 W}{\partial y^2} = f(x) \quad \text{at } y = 0 \quad \text{for } 0 < x < 1,\\
&\text{(e)} \quad \frac{\partial^2 W}{\partial y^2} = g(x) \quad \text{at } y = 1 \quad \text{for } 0 < x < 1,
\end{aligned}
$$

where R is the open region $(0 < x < 1, 0 < y < 1)$ and $f(x)$ and $g(x)$ describe the bending moments.

We superimpose a square mesh over the region R with grid size $h = \frac{1}{M}$, where M is a positive integer, and employ the following notation:

$$
\begin{aligned}
&w(ih, jh) = w_{i,j},\\
&\nabla_x^2 w_{i,j} = w_{i+1,j} - 2w_{i,j} + w_{i-1,j},
\end{aligned}
$$

and

$$
\nabla_x^4 w_{i,j} = w_{i+2,j} - 4w_{i+1,j} + 6w_{i,j} - 4w_{i-1,j} + w_{i-2,j}.
$$

The usual finite difference approximations to (2) are:

$$
\begin{aligned}
&\text{(a)} \quad \nabla_x^4 w_{i,j} + 2\nabla_y^2 \nabla_x^2 w_{i,j} + \nabla_y^4 w_{i,j} = 0 \quad (ih, jh) \in R',\\
&\left.\begin{aligned}
&\text{(b)} \quad w_{i,j} = 0\\
(3)\quad &w_{i+1,j} = -w_{i-1,j} \quad (i = 0, M)
\end{aligned}\right\} \quad (j = 1, 2, \cdots M - 1),\\
&\text{(c)} \quad w_{i,0} = w_{i,M} = 0 \qquad (i = 1, 2, \cdots M - 1),\\
&\text{(d)} \quad w_{i,1} = -w_{i,-1} + h^2 f(ih) \qquad (i = 1, 2, \cdots M - 1),\\
&\text{(e)} \quad w_{i,M+1} = -w_{i,M-1} + h^2 g(ih) \qquad (i = 1, 2, \cdots M - 1),
\end{aligned}
$$

where R' is the set of lattice points (ih, jh) in R.

Since formulation (3) represents a system of $(M-1)^2$ equations in $(M-1)^2$ unknowns, the amount of computing time involved in solving (3) will be impractical for large M unless one can employ a numerical method which avoids the use of $(M-1)^2$th order systems. Even when the usual point relaxation methods are used, they converge very slowly. Thus for the solution of (3) we consider the following iteration scheme which involves line rather than point relaxation:

$$
\begin{aligned}
&\text{(a)}\quad u_{i,j}^{(n+\frac{1}{2})} = u_{i,j}^{(n)} - r_{n+1}\nabla_x^4 u_{i,j}^{(n+\frac{1}{2})} - 2r_{n+1}\nabla_y^2\nabla_x^2 u_{i,j}^{(n)} - r_{n+1}\nabla_y^4 u_{i,j}^{(n)},\\
&\text{(b)}\quad u_{i,j}^{(n+1)} = u_{i,j}^{(n+\frac{1}{2})} - r_{n+1}\nabla_y^4 u_{i,j}^{(n+1)} + r_{n+1}\nabla_y^4 u_{i,j}^{(n)} \quad (ih, jh) \in R',\\
&\text{(c)}\quad \left.\begin{aligned} u_{i,j}^{(n)} &= w_{i,j} = 0\\ u_{i+1,j}^{(n)} &= -u_{i-1,j}^{(n)} \quad (i = 0, M)\end{aligned}\right\} \quad (j = 1, 2, \cdots M-1),\\
&\text{(d)}\quad u_{i,j}^{(n)} = w_{i,j} = 0 \quad (j = 0, M) \qquad (i = 1, 2, \cdots M-1),\\
&\text{(e)}\quad \left.\begin{aligned} u_{i,1}^{(n)} &= -u_{i,-1}^{(n)} + h^2 f(ih)\\ u_{i,M+1}^{(n)} &= -u_{i,M-1} + h^2 g(ih)\end{aligned}\right\} \quad (i = 1, 2, \cdots M-1).
\end{aligned}
\tag{4}
$$

Here r_{n+1} is an iteration parameter which will be chosen to accelerate convergence and $u_{i,j}^{(0)}$, for (ih, jh) in R', is an initial approximation to $w_{i,j}$.

Equation 4(a) is implicit in x alone and represents a system of $M-1$ equations in $M-1$ unknowns along a single line in the x-direction. A similar statement can be made for equation 4(b) and the variable y. Hence, the numerical procedure for executing one iteration consists of first solving 4(a) for $u_{i,j}^{(n+\frac{1}{2})}$ at all points in R' and subsequently solving 4(b) for $u_{i,j}^{(n+1)}$. At every stage of this iteration scheme, the matrices involved are all of order $M-1$ with at most 5 non-zero elements in any row (either on the main diagonal or on two adjacent diagonals). Such "quidiagonal" systems can be solved efficiently by an extension of an algorithm for solving tridiagonal matrices due to L. H. Thomas and will be discussed later in Section 5.

3. Convergence of the iterative procedure. To study the convergence of $u_{i,j}^{(n)}$ to $w_{i,j}$, it is convenient to eliminate $u_{i,j}^{(n+\frac{1}{2})}$ from 4(a) and 4(b). This yields the single equation

$$u_{i,j}^{(n+1)} = u_{i,j}^{(n)} - r_{n+1}(\nabla_x^4 u_{i,j}^{(n+1)} + \nabla_y^4 u_{i,j}^{(n+1)} + 2\nabla_y^2\nabla_x^2 u_{i,j}^{(n)}) - r_{n+1}^2\nabla_x^4\nabla_y^4(u_{i,j}^{(n+1)} - u_{i,j}^{(n)}).$$

Now let the error $e_{i,j}^{(n)}$ be defined by

$$e_{i,j}^{(n)} = u_{i,j}^{(n)} - w_{i,j}.$$

Then $e_{i,j}^{(n)}$ satisfies the following equation.

$$
\begin{aligned}
(5)\quad e_{i,j}^{(n+1)} = e_{i,j}^{(n)} &- r_{n+1}(\nabla_x^4 e_{i,j}^{(n+1)} + \nabla_y^4 e_{i,j}^{(n+1)} + 2\nabla_y^2\nabla_x^2 e_{i,j}^{(n)})\\
&- r_{n+1}^2\nabla_x^4\nabla_y^4(e_{i,j}^{(n+1)} - e_{i,j}^{(n)}), \quad (ih, jh) \in R'
\end{aligned}
$$

together with homogeneous boundary conditions corresponding to relations 4(c), 4(d) and 4(e).

The eigenfunctions of (5) are of the form

$$e_{i,j}^{(n)} = A \sin(p\pi ih) \sin(q\pi jh).$$

Now suppose

$$e_{i,j}^{(n+1)} = \lambda_{p,q} e_{i,j}^{(n)}, \quad p, q = 1, 2, \cdots M-1.$$

By substituting this expression into (5) we find that

$$\lambda_{p,q}(\alpha_{n+1}) = \frac{(1 - \alpha_{n+1}S_p^2S_q^2)^2}{1 + \alpha_{n+1}(S_p^4 + S_q^4) + \alpha_{n+1}^2S_p^4S_q^4}, \tag{6}$$

where we have set

$$\alpha_{n+1} = 16r_{n+1},$$

$$S_p = \sin\frac{p\pi h}{2}$$

and

$$S_q = \sin\frac{q\pi h}{2}.$$

Since the initial error $e_{i,j}^{(0)}$ can be expanded in the form

$$e_{i,j}^{(0)} = \sum_{p,q=1}^{M-1} C_{p,q}^{(0)} \sin(p\pi ih) \sin(q\pi jh)$$

we have

$$e_{i,j}^{(n)} = \sum_{p,q=1}^{M-1} C_{p,q}^{(n)} \sin(p\pi ih) \sin(q\pi jh)$$

where

$$C_{p,q}^{(n)} = \prod_{k=1}^{n} \lambda_{p,q}(\alpha_k) C_{p,q}^{(0)}.$$

For all positive α_k, it is clear from (6) that $0 \leq \lambda_{p,q}(\alpha_k) < 1$ for all p and q. Thus, after n iterations each component of the error decreases by a factor of

$$\prod_{k=1}^{n} \lambda_{p,q}(\alpha_k)$$

as long as α_k (and hence r_k) is positive. The minimum value of $\lambda_{p,q}(\alpha)$ is zero and occurs when

$$\alpha = \frac{1}{S_p^2S_q^2}.$$

This implies, at least theoretically, that after $\frac{M(M-1)}{2}$ iterations, with appropriate choices for the α_k, one could make all components of the error vanish and thus obtain the exact solution. In practice, however, it is more efficient to reduce all components of the error by a fraction of their original value. With this in mind, we seek a method for choosing the α_k which optimizes the convergence rate of the iterative procedure.

Let

$$\bar{\lambda}_{p,q}(\alpha) \equiv \left(\frac{1 - \alpha S_p^2S_q^2}{1 + \alpha S_p^2S_q^2}\right)^2.$$

Since $S_p^4 + S_q^4 \geq 2S_p^2S_q^2$ it follows from (6) that

$$\lambda_{p,q}(\alpha) \leq \bar{\lambda}_{p,q}(\alpha).$$

In view of this inequality, any convergence results based on $\bar{\lambda}_{p,q}(\alpha)$ will also apply when $\bar{\lambda}_{p,q}(\alpha)$ is replaced by $\lambda_{p,q}(\alpha)$. In what follows we shall base our considerations on the more convenient quantity $\bar{\lambda}_{p,q}$.

4. Determination of the iteration parameters. We next consider the problem of finding a set of iteration parameters α_k $(k = 1, 2, \cdots t)$ such that the maximum of the function

$$G_t(S_p^2S_q^2, \alpha_k) = \prod_{k=1}^{t} \bar{\lambda}_{p,q}(\alpha_k) = \prod_{k=1}^{t} \left(\frac{\alpha_k^{-1} - S_p^2S_q^2}{\alpha_k^{-1} + S_p^2S_q^2} \right)^2$$

is a minimum for p and q in the range $1, 2, \cdots M-1$. This problem may be resolved efficiently by treating it as a Chebyshev minimax problem and is discussed in [4]. Applying the results obtained there to the function $G_t(S_p^2S_q^2, \alpha_k)$ we find that a recommended set of iteration parameters is

$$\alpha_k = x^{1-k}, \quad k = 1, 2, \cdots t \tag{7}$$

for $0 < x < 1$. This gives an upper bound on $G_t(S_p^2S_q^2, \alpha_k)$ defined by

$$G_t(S_p^2S_q^2, \alpha_k) \leq P_t(x) \equiv \left(\frac{1 - x^t}{1 + x^t} e^{-\frac{x^t}{1-x}} \right)^4 \tag{8}$$

provided that

$$x^{t-1} \leq S_1^4 \quad \text{and} \quad \frac{S_1^4}{S_{M-1}^4} \ll 1. \tag{9}$$

The iteration parameters (7) are not asserted to be the best possible set. A measure of the loss of convergence due to not using the set of optimum parameters $\bar{\alpha}_k$ $(k = 1, 2, \cdots, t)$ is given by

$$\frac{G_t(S_p^2S_q^2, \bar{\alpha}_k)}{G_t(S_p^2S_q^2, \alpha_k)} \approx e^{-\frac{2x^t}{1-x}}. \tag{10}$$

The formal process of choosing the iteration parameters α_k proceeds as follows: First one decides on a value of $P_t(x)$. When the initial error is to be reduced by a given factor, the choice of $P_t(x)$ determines the number of cycles necessary for this reduction. Next, x is determined from (8) while the number t of iterations per cycle is computed by solving (9) for t; i.e.,

$$t \geq 1 + \frac{4 \log S_1}{\log x}. \tag{11}$$

Hence, after one cycle, consisting of t double sweeps, every component of the initial error will be reduced by a factor of $P_t(x)$.

The significance of relation (10) is important for the initial selection of $P_t(x)$ and hence x. In order to carry out the iteration procedure in the most efficient manner, the value of x should be small enough so that this ratio is close to unity. Although a decrease in x means an increase in $P_t(x)$, it still may be more efficient in terms of total sweeps, to increase the number of cycles while at the same time

decreasing the number of iterations within a cycle. At any rate, for this scheme to give the best results, $P_t(x)$ should be such that $x < 0.2$. For this range

$$\frac{G_t(S_p^2 S_q^2, \bar{\alpha}_k)}{G_t(S_p^2 S_q^2, \alpha_k)} > 0.8.$$

As an example of the method, consider the solution of system (4) using a 20×20 grid. Suppose we desire to reduce the initial error by a factor of 10^{-6}. We choose $x = 0.2$ so that $P_t(x) \simeq 0.01$. This means that 3 cycles are necessary for the reduction. The number of double sweeps per cycle is given by

$$t \sim 1 + \frac{4 \log \sin \frac{\pi}{40}}{\log .2} \simeq 7.35.$$

Hence, it takes three cycles of eight iterations each, using

$$r_k = \frac{(0.2)^{1-k}}{16}, \quad k = 1, 2, \cdots 8$$

to reduce the error by approximately 10^{-6}. This involves 48 single sweeps over the grid.

5. The solution of the quidiagonal systems. The quidiagonal systems arising from the iteration procedure (4) may be written in the general form:

$$(12) \quad \begin{bmatrix} C_1 & D_1 & E_1 & & & & & \\ B_2 & C_2 & D_2 & E_2 & & & & \\ A_3 & B_3 & C_3 & D_3 & E_3 & & & \\ & A_4 & B_4 & C_4 & D_4 & E_4 & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & & A_{M-2} & B_{M-2} & C_{M-2} & D_{M-2} & \\ & & & & A_{M-1} & B_{M-1} & C_{M-1} & \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_{M-1} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ F_{M-1} \end{bmatrix}.$$

Such systems may be solved by the following elimination scheme: Let

$$\begin{aligned} &\omega_1 = C_1, \\ &\beta_1 = D_1/\omega_1, \quad \beta_0 = 0, \qquad \beta_{M-1} = 0, \\ &\gamma_1 = E_1/\omega_1, \quad \gamma_0 = 0, \quad \gamma_{M-1} = \gamma_{M-2} = 0, \end{aligned}$$

and define recursively

$$\begin{aligned} \delta_j &= B_j - A_j\beta_{j-2}, \\ \omega_j &= C_j - A_j\gamma_{j-2} - \delta_j\beta_{j-1}, \quad \text{for } 2 \le j \le M-1, \\ \beta_j &= (D_j - \delta_j\gamma_{j-1})/\omega_j \\ \gamma_j &= E_j/\omega_j. \end{aligned}$$

Next form

$$\begin{aligned} h_0 &= 0, \\ h_1 &= F_1/\omega_1, \\ h_j &= (F_j - A_j h_{j-2} - \delta_j h_{j-1})/\omega_j, \quad 2 \le j \le M-1. \end{aligned}$$

The values of u are then obtained successively from the formulas

$$u_{M-1} = h_{M-1},$$
$$u_j = h_j - \beta_j u_{j+1} - \gamma_j u_{j+2}, \quad j = M-2, M-3, \cdots 1.$$

For the matrices arising from equations 4(a) and 4(b) we have

$$\begin{aligned} B_j &= D_j = -4 \quad \text{for all } j, \\ A_j &= E_j = 1 \quad \text{for all } j, \end{aligned} \tag{13}$$

while the diagonal elements depend on the iteration parameters r_k. The components F_j on the right hand side of the general system (12) will, in our case, involve values of u obtained on the previous iteration and may be evaluated in a straightforward manner. When relaxing along lines in the y-direction or along either of the lines $y = h, y = (M-1)h$, these components will also require values of $f(x)$ and $g(x)$. For one complete iteration (i.e., a double sweep over the grid) one must solve $2(M-1)$ systems of the form (12) with each system of order $M-1$. Using the elimination scheme described above, this takes approximately $30(M-1)^2$ additions, $13(M-1)^2$ multiplications and $6(M-1)^2$ divisions. In arriving at this estimate, we have made use of relations (13). Regarding both a multiplication and a division as equivalent timewise to 5 additions, we may estimate the total time T for one iteration as $25(M-1)^2$ where τ is the average time in seconds for one multiplication. For a computer capable of executing 4000 multiplications per second we have approximately

$$T = 6.25 \times 10^{-3} (M-1)^2 \text{ seconds.} \tag{14}$$

Thus, in the example at the end of Section 4, it would take approximately 54 seconds to reduce the initial error by a factor of 10^{-6}.

6. Remarks. The particular fourth order problem considered here can, of course, be solved by more direct methods. The second-order Richardson method seems to have been successfully used although it has recently been shown by E. Windsor [5] that the unextrapolated Richardson Method does not converge. Another successful approach has been to reformulate the fourth order equation as two coupled second order elliptic equations which can then be solved by standard methods.

On the basis of the theory presented here, the alternating direction method seems to be decidedly superior to other proposed relaxation methods, the factor of superiority generally increasing as the number of mesh points increases.

The method proposed here for solving the biharmonic equation can readily be adapted to the case of rectangular regions with other boundary conditions. However, for arbitrary regions and for general boundary conditions, the proof given here does not hold and hence convergence is not assured.

Digital Computing Center
The Ramo-Wooldridge Corporation
P. O. Box 45564, Airport Station
Los Angeles 45, California

1. J. Douglas, Jr. & H. H. Rachford, Jr., "On the numerical solution of heat conduction problems in two and three space variables," Amer. Math. Soc., *Trans.*, v. 82, 1956, p. 421–439.
2. M. Held, J. Heller, & D. Lubell, "Rates of convergence of successive iteration schemes for fourth order elliptic difference equations," New York University. (To be published).

3. S. P. FRANKEL, "Convergence rates of iterative treatments of partial differential equations," *MTAC*, v. IV, 1950, p. 65–75.
4. E. L. WACHSPRESS, "Cure: a generalized two space-dimension multigroup coding for the IBM 704," KAPL-1724, AEC Research and Development Report, Physics and Math. (TID-4500, 13 Ed.), Knolls Atomic Power Lab., Schenectady, N. Y.
5. E. WINDSOR, "Iterative solutions of biharmonic differential equations," New York University, Master's Thesis, May, 1957.

TECHNICAL NOTES AND SHORT PAPERS

On The Numerical Evaluation of Cauchy Principal Values of Integrals

By I. M. Longman

Introduction. This note demonstrates a simple method for the numerical evaluation of the Cauchy principal value of an integral

$$P\int_a^b f(x)dx \quad (x \text{ real}),$$

when the integrand $f(x)$ has singularities at one or more points in the finite interval $[a, b]$ of integration. The necessity for such numerical evaluation arises sometimes in the solution of problems in applied mathematics. Such integrals can occur, for example, in the solution of wave propagation problems. For instances of this see Lamb [1], and Pekeris [2]. Sometimes a singularity can be avoided by a suitable transformation in the complex plane. The present note presents a simple direct method of dealing with such a singularity.

Formulation of the method. The method is quite general and can be applied separately to each singularity of $f(x)$, and so, for simplicity, we will suppose that $f(x)$ has only one singularity which, without loss of generality, we may suppose to be at $x = 0$. Suppose then we require to evaluate by some numerical quadrature method the integral

$$I = P\int_{-a}^{a} f(x)dx \tag{1}$$

where the range of integration is made symmetrical about the origin by splitting up the integral into two ranges if necessary. We split up the function $f(x)$ into its odd and even components

$$f(x) = \tfrac{1}{2}[f(x) - f(-x)] + \tfrac{1}{2}[f(x) + f(-x)] = g(x) + h(x), \text{ say.} \tag{2}$$

Then

$$I = P\int_{-a}^{a} g(x)dx + P\int_{-a}^{a} h(x)dx. \tag{3}$$

Now since $g(x)$ is an odd function the first integral vanishes, and so

$$I = 2\int_0^a h(x)dx, \tag{4}$$

where either there is now no singularity at $x = 0$, or, if there is, it is not such as to make the integral (4) diverge, since we suppose the Cauchy principal value

Received 18 March 1958. This work was done while the author was working at The Weizmann Institute of Science, Rehovot, Israel.

(1) to exist. If (4) still contains a singularity (e.g. of the form $\log x$) the problem is no longer that of finding a Cauchy principal value, but the entirely different one of the numerical evaluation of an infinite integral. In such a case the singularity in (4) can usually be removed by a substitution of the form $x = y^n$, for a sufficiently large value of n. But, in general, such a singularity arises from the presence of a pole on the path of integration and this must be of odd order if the Cauchy principal value is to exist. In this case the numerical evaluation has been reduced to the determination of (4) which no longer contains a singularity. However, in (4), at least in the first instance, the integrand may reduce to the indeterminate form $\infty - \infty$ or equivalently $0/0$, at the lower limit of integration, but this is not a serious difficulty. In the first place it may be that $\lim_{x \to 0} h(x)$ is known (as in examples 1 and 2), or it may be possible to eliminate the indeterminate form by algebraic manipulation as in example 3. In any case if $h(x)$ or at least the part of it which gives the indeterminateness is an analytic function, values at and near $x = 0$ can be obtained by expansion. Another way of overcoming the difficulty is the use an integration formula, e.g. Gaussian [3], [4], which does not involve the end points of the range of integration.

Examples. The method is illustrated by three examples.

(1) To evaluate

$$I = P \int_{-1}^{1} (e^x/x)dx.$$

We have

$$I = P \int_{-1}^{1} (1/x) \cosh x \, dx + P \int_{-1}^{1} (1/x) \sinh x \, dx = 2 \int_{0}^{1} (1/x) \sinh x \, dx$$

which can now be evaluated numerically with no disturbing singularity.

(2) To evaluate

$$I = P \int_{0}^{\pi/2} \frac{e^x}{\sin x - \cos x} dx.$$

Here the singularity is at $x = \pi/4$, and making the substitution $x = \pi/4 + y$ a little manipulation shows that the integral is reduced to

$$I = 2^{\frac{1}{2}} e^{\pi/4} \int_{0}^{\pi/4} (\sinh y)/(\sin y)dy.$$

(3) To evaluate

$$I = P \int_{-\pi/2}^{\pi/2} \frac{dy}{\sqrt{1 + \sin y} - 1}.$$

The singularity is at $y = 0$, and proceeding on the same lines as above we obtain the form

$$I = 2 \int_{0}^{\pi/2} \frac{dy}{\sqrt{1 - \sin y} + \sqrt{1 + \sin y}}$$

which does not contain any singularities.

Institute of Geophysics
University of California
Los Angeles, California

1. H. LAMB, "On the propagation of tremors over the surface of an elastic solid," Roy. Soc. London, *Phil. Trans.*, ser. A, v. 203, 1904.
2. C. L. PEKERIS, "The seismic surface pulse," Nat. Acad. Sci., *Proc.*, v. 41, 1955, p. 469–480.
3. NBS Applied Mathematics Series, 37, *Tables of Functions and of Zeros of Functions*, U. S. Gov. Printing Office, Washington, D. C., 1954. [*MTAC*, v. 10, 1956, Rev. 104, p. 249–250.]
4. P. DAVIS & P. RABINOWITZ, "Abscissas and weights for Gaussian quadratures of high order," National Bureau of Standards, *Jn. of Research*, v. 56, 1956, p. 35–37. [*MTAC*, v. 11, 1957, Rev. 84.]

Mersenne Numbers

By Hans Riesel

During 1957 the author of this note had the opportunity of running the Swedish electronic digital computer BESK in order to examine Mersenne numbers. The intention of the author's investigation on the BESK was to check some known results, and to examine some Mersenne numbers not previously examined.

Mersenne numbers are numbers $M_p = 2^p - 1$, where p is a prime. See [1], which contains a more complete list of references. The Mersenne numbers have attained interest in connection with digital computers because there is a simple test to decide whether they are prime or composite. This is Lucas's test [1]. Furthermore the number $2^{p-1}M_p$ is a perfect number, if M_p is a prime, and all known perfect numbers are of this form.

In the beginning of 1957 a program for testing the primeness of the Mersenne numbers on the BESK was worked out by the author. This program, using Lucas's test, works for all $p < 10000$. As a test, this program was run for the following values of p: 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, and 2281. The result was that the numbers M_p are prime for these values of p, thus confirming known results.

After these tests, values of $p > 2300$ were to be tested. These values had not been tested before. But since the testing of M_p for one value of p of this order takes several hours on the BESK, a special program for calculating the smallest factor of M_p, if this factor is $< 10 \cdot 2^{20} = 104\ 85760$, was worked out. This special program is based on the following well-known theorem:

All prime factors q of M_p $(p > 2)$ are of the form $q = 2kp + 1$, and of one of the two forms $q = 8l \pm 1$.

The proof of the theorem is quite simple. If q is a factor of M_p, $2^p \equiv 1 \pmod{q}$. Since p is a prime, since all numbers n for which $2^n = 1 \pmod{q}$ form a module, and since $2^p \equiv 1 \pmod{q}$, this module consists of all integral multiples of the prime p. Now $2^{q-1} \equiv 1 \pmod{q}$ if q is a prime, and hence $q - 1$ is a multiple of p, and in fact an even multiple (since q must be an odd number). This is the first part of the theorem: $q - 1 = 2kp$ $(k = 1, 2, 3, \cdots)$. The second part follows immediately from the theory of quadratic residues. Since $x^2 \equiv 2 \pmod{q}$ has the solution $x \equiv 2^{(q+1)/2} \pmod{q}$, we see that 2 is a quadratic residue mod q, hence $q \equiv \pm 1 \pmod{8}$.

By the above mentioned special program for small factors of M_p the values of $M_p \pmod{q}$ for all primes q of the theorem were now calculated. When this residue was $\equiv 0$, the factor q was printed out. When no factor $q < 10 \cdot 2^{20}$ was found, the BESK turned to the next value of p. This program was run for all

Received 8 November 1957.

values of $p < 10000$. The running time on the BESK for one value of p lay between 0 and 2 minutes, depending on the size of the smallest factor. These smallest factors are given below in a table. The known Mersenne primes are also included in the table. Finally, Cole's factor of M_{67}, and Robinson's factors of M_{109} and M_{157}, though greater than $10 \cdot 2^{20}$, have been printed in the table. On checking the values against other sources, a misprint in Archibald's note [2] was detected. This misprint, which concerns the value of the smallest factor of M_{163}, seems to have come from Kraitchik [3], p. 24 and p. 92. As a further check, all factors in our table have been looked up in Lehmer's Prime Tables, except a few, which are too large, and a separate calculation was made, to check that they are of the form $2kp + 1$. Since, however, there are disturbances in digital computers, it is not absolutely sure that all these numbers really are factors of the corresponding Mersenne numbers M_p. Those primes p, for which no factor $< 10 \cdot 2^{20}$ of M_p was found, are, except the known Mersenne primes, omitted in the table.

When this table had been calculated, the BESK examined the omitted values of p with Lucas's test. This was done for all values of p between 2300 and 3300. Since the available running time for testing Mersenne numbers on the BESK was limited, every value of p was tested only once. The final remainder, see [1], was printed out in hexadecimal form. On September 8th, 1957, a run indicated that the number M_{3217} is prime. This result was repeated on September 12th. All other numbers tested turned out to be composite; however, this result could be false, since the running time is very long. The testing of M_{3217} took about $5^h\ 30^m$ on the BESK, for one run.

A table of the smallest factor of $2^p - 1$, p prime follows. Primes $2^p - 1$ are indicated by a dash.

Ormangsgatan 67C
Stockholm-Vallingby, Sweden

1. R. M. Robinson, "Mersenne and Fermat numbers," Amer. Math. Soc., *Proc.*, v. 5, p. 842–846.
2. R. C. Archibald, "Mersenne's numbers," *Scripta Mathematica*, v. 3, 1935, p. 112–119.
3. M. Kraitchik, *Recherches sur la Theorie des Nombres*, Gauthier-Villars, Paris, 1952.

Table of Factors of $2^p - 1$:

p	factor	p	factor	p	factor	p	factor	p	factor	p	factor
2	—	3	—	5	—	7	—	11	23	13	—
17	—	19	—	23	47	29	233	31	—	37	223
41	13367	43	431	47	2351	53	6361	59	1 79951	61	—
67	1937 07721	71	2 28479	73	439	79	2687	83	167	89	—
97	11447	107	—	109	7459 88807	113	3391	127	—	131	263
151	18121	157	8521 33201	163	1 50287	167	23 49023	173	7 30753	179	359

Table of Factors of $2^p - 1$: $\frac{p}{\text{factor}}$—*Continued*

181	191	197	211	223	229
43441	383	7487	15193	18287	15 04073
233	239	251	263	277	281
1399	479	503	23671	11 21297	80929
283	311	317	337	353	359
9623	53 44847	9511	18199	9 31921	719
367	383	397	419	431	443
12479	14 40847	2383	839	863	887
449	461	463	487	491	499
12 56303	2767	11113	4871	983	20959
521	547	557	571	577	587
—	5471	3343	5711	3463	5 54129
593	601	607	617	619	641
1 04369	3607	—	59233	1 10183	35897
643	659	683	701	719	743
31 89281	1319	1367	7 96337	1439	1487
757	761	773	811	827	829
98 15263	4567	68 64241	3 26023	66161	72953
839	857	859	863	877	881
26849	6857	72 15601	82 58911	35081	26431
883	907	911	929	937	941
8831	11 70031	1823	13007	28111	7529
953	967	977	1009	1013	1019
3 43081	23209	8 67577	34 54817	6079	2039
1021	1031	1033	1039	1049	1051
40841	2063	1 96271	50 80711	33569	35 75503
1091	1093	1097	1103	1117	1129
87281	43721	9 80719	2207	53617	33871
1153	1181	1187	1193	1201	1213
2 67497	47 42897	2 56393	1 21687	57649	3 27511
1223	1229	1231	1249	1279	1303
2447	36871	5 31793	97423	—	104 44849
1321	1327	1361	1367	1381	1423
7927	27 30967	8167	10937	8287	16 99063
1433	1439	1447	1451	1453	1459
20063	2879	57881	2903	8719	14591
1481	1489	1499	1511	1523	1531
71089	71473	2999	3023	25 22089	88799
1543	1559	1583	1607	1609	1663
1 01839	3119	3167	28927	23 94193	16631
1667	1693	1697	1709	1721	1723
13337	10159	12 35417	3 79399	1 75543	17231

Table of Factors of $2^p - 1$: $\frac{p}{\text{factor}}$—*Continued*

1741	1777	1789	1801	1811	1823
10 02817	10663	39359	28817	3623	1 20319
1847	1861	1871	1877	1879	1913
33247	10 23551	14969	15017	6 05039	63 01423
1931	1973	1987	1993	1997	1999
3863	1 22327	67559	11959	3 95407	18 07097
2003	2011	2017	2039	2063	2069
4007	21 71881	93 38711	4079	4127	3 26903
2081	2083	2111	2113	2131	2141
2 66369	72 69671	3 41983	22 31329	32 73217	3 89663
2179	2203	2207	2213	2251	2281
2 48407	—	1 23593	53113	4 00679	—
2297	2333	2339	2351	2389	2393
57 65471	3 12623	4679	4703	71671	33503
2399	2411	2417	2447	2459	2531
4799	19289	14503	81 63193	4919	7 89673
2539	2543	2549	2591	2593	2609
25391	5087	43 63889	7 66937	15559	36527
2617	2621	2657	2663	2677	2687
78511	15727	37199	63913	3 64073	1 98839
2689	2699	2707	2711	2741	2753
71 58199	5399	1 73249	5 85577	82231	47 95727
2767	2819	2837	2843	2857	2897
6 25343	5639	22697	1 42151	36 79817	17383
2903	2909	2939	2953	2963	2999
5807	1 10543	5879	88591	5927	6 71777
3001	3011	3023	3037	3041	3119
32 17073	20 23393	6047	18223	24329	24953
3121	3137	3163	3181	3217	3221
11 23561	20 01407	4 55473	19087	—	6 44201
3257	3299	3319	3323	3329	3347
97711	6599	33191	23 12809	26633	26777
3359	3361	3391	3433	3457	3491
6719	5 57927	2 98409	44 42303	20743	6983
3499	3511	3517	3527	3529	3533
35 82977	35111	5 62721	63487	1 05871	1 90783
3539	3541	3557	3571	3581	3593
7079	7 64857	4 26841	64279	21487	21559
3623	3659	3701	3719	3733	3767
7247	41 05399	1 11031	43 51231	13 14017	30137
3779	3793	3797	3803	3823	3851
7559	60689	91129	7607	89 68759	7703

Table of Factors of $2^p - 1$: $\frac{p}{\text{factor}}$—*Continued*

3853	3863	3907	3911	3917	3923
33 90641	7727	5 31353	7823	4 07369	2 19689
3931	3967	3989	4001	4013	4019
26 18047	63473	1 91473	24007	1 20391	8039
4057	4073	4099	4127	4129	4153
97369	56 69617	73783	74287	15 85537	91367
4211	4217	4229	4243	4271	4273
8423	21 84407	3 29863	1 01833	8543	25639
4289	4297	4373	4391	4441	4447
34313	8 93777	61223	8783	26647	71153
4457	4483	4513	4517	4549	4561
16 31263	58 54799	1 35391	27103	1 36471	72977
4597	4603	4639	4649	4657	4691
27583	6 26009	1 94839	5 95073	11 45623	5 72303
4723	4759	4787	4793	4801	4813
29 09369	1 14217	1 14889	67103	28807	28879
4861	4871	4903	4919	4933	4943
29167	9743	49031	9839	29599	9887
4957	4967	4993	4999	5003	5011
1 48711	3 27823	79889	2 09959	10007	80177
5021	5039	5051	5087	5101	5107
40169	10079	10103	40697	81 10591	51071
5167	5171	5179	5189	5197	5227
65 31089	10343	2 48593	1 55671	31183	11 29033
5231	5233	5237	5261	5279	5297
10463	9 94271	40 22017	42089	10559	44 49481
5303	5347	5393	5399	5413	5417
10607	3 10127	32359	10799	53 26393	4 33361
5431	5437	5441	5449	5483	5501
54311	20 87809	36 56353	25 17439	8 88247	6 93127
5519	5521	5531	5557	5563	5623
6 62281	88337	9 40271	33343	5 34049	11 35847
5639	5683	5711	5717	5743	5779
11279	8 75183	11423	34303	5 43217	9 24641
5791	5801	5807	5843	5849	5861
9 26561	34807	1 39369	7 12847	41 05999	46889
5903	5939	6011	6029	6067	6101
11807	6 17657	2 88529	84407	2 06279	79 06897
6113	6131	6143	6163	6173	6197
25 30783	12263	8 84593	5 91649	37039	2 97457
6199	6203	6263	6277	6287	6311
61991	1 48873	12527	37663	50 29601	36 47759

Table of Factors of $2^p - 1$: $\frac{p}{\text{factor}}$—*Continued*

6317	6323	6343	6353	6367	6373
6 94871	12647	90 45119	38119	63671	38239
6389	6397	6421	6449	6473	6481
1 91671	19 44689	56 89007	51593	27 96337	6 22177
6491	6521	6529	6551	6553	6563
12983	48 90751	7 31249	13103	14 54767	13127
6569	6607	6637	6673	6691	6703
66 21553	80 34113	52 03409	67 66423	3 21169	4 42399
6719	6737	6781	6803	6841	6871
12 90049	7 41071	1 08497	2 17697	41047	2 88583
6883	6899	6917	6947	6949	6959
18 85943	13799	55337	1 80623	68 79511	55673
6967	6977	6983	6997	7039	7043
10 72919	41863	13967	21 69071	12 52943	14087
7057	7079	7103	7129	7151	7193
6 21017	14159	14207	18 67799	14303	54 09137
7207	7211	7297	7411	7417	7459
1 72969	14423	18 68033	10 22719	1 18673	1 34263
7487	7529	7537	7561	7573	7589
1 79689	1 05407	7 23553	5 89759	45439	2 88383
7591	7639	7643	7673	7681	7687
18 97751	5 50009	15287	1 84153	39 94121	4 45847
7691	7741	7759	7789	7823	7841
15383	89 64079	1 39663	8 72369	15647	23 67983
7853	7873	7883	7901	7919	8011
47119	11 80951	15767	47407	63353	80111
8017	8059	8101	8111	8161	8167
1 28273	70 43567	7 12889	16223	67 08343	31 36129
8171	8179	8219	8221	8231	8237
8 00759	43 18513	19 88999	9 20753	2 14007	10 54337
8243	8269	8273	8293	8317	8353
16487	7 27673	10 42399	1 99033	3 82583	50119
8377	8387	8419	8423	8429	8431
50263	92 76023	4 88303	3 53767	67433	8 09377
8461	8467	8501	8513	8597	8599
14 04527	2 03209	6 80081	2 72417	4 81433	85991
8629	8663	8669	8693	8699	8713
2 58871	17327	26 18039	2 60791	4 17553	16 03193
8719	8731	8737	8741	8761	8783
28 77271	25 14529	6 81487	69929	1 92743	2 28359
8803	8807	8821	8839	8867	8929
28 34567	2 28983	4 05767	65 23183	70937	1 96439
8951	8963	9001	9007	9029	9059
17903	6 63263	15 12169	90071	7 22321	18119

Table of Factors of $2^p - 1$: $\underset{\text{factor}}{p}$—*Continued*

9109	9127	9137	9157	9173	9181
49 55297	1 46033	10 05071	30 76753	4 95343	13 77151
9221	9283	9311	9323	9337	9341
7 19239	59 41121	14 15273	8 95009	26 14361	74729
9343	9371	9391	9397	9403	9419
1 49489	18743	93911	2 25529	25 76423	18839
9421	9431	9461	9479	9491	9497
85 73111	6 79033	75689	18959	5 31497	5 31833
9511	9521	9539	9601	9613	9619
95111	3 61799	19079	22 85039	57679	6 15617
9791	9811	9829	9833	9851	9859
19583	77 70313	7 07689	45 62513	78809	11 04209
9883	9949	9973			
1 58129	80 98487	2 99191			

A Computation of Some Bi-Quadratic Class Numbers

By Harvey Cohn

A fascinating chapter in computational number theory began when Lagrange showed that every positive integer is representable as the sum of at most *four* perfect squares [1]. Clearly three would not suffice in every case, as $7 = 2^2 + 1^2 + 1^2 + 1^2$ would be an exception; nevertheless, the problem of expressing some positive integer n as the sum of at most *three* squares soon achieved a very special role. For, Gauss showed that $r(n)$, the number of such representations, is connected in a very simple way with the much studied (intrinsically positive) class number, h, of the field generated by $\sqrt{-n}$. Specifically for n square-free (and $n \neq 1, 3$ where $h = 1$),

$$r(n) = gh \tag{1}$$

where $g = 12$ for $n \equiv 1, 2, 5, 6$ and $g = 24$ for $n \equiv 3 \pmod 8$. Thus we could conclude the existence of at least one such representation for the indicated n. Gauss and later, Kronecker, reversed the direction of these equations by making large scale tabulations of h from $r(n)$, although, unfortunately, no location for Kronecker's alleged tabulation (for odd n up to 10,000) seems to exist in the literature. In tallying the representation $n = x_1^2 + x_2^2 + x_3^2$ it might be noted that one must count each ordered triple (x_1, x_2, x_3) of positive, negative, or zero integers as a separate unit, so that as much as $2^3 \cdot 3! = 48$ could be contributed to $r(n)$ when such a decomposition into squares is expressed as triples.

In more recent times, the representation theory was extended to integers in the field k generated by $\sqrt{5}$, i.e., to the quantities $\mu = (a + b\sqrt{5})/2$ where a and b are of the same parity. Here we seek to represent, necessarily, only those integers μ which are positive together with their conjugate (i.e., totally positive). Thus, e.g., $a > |b\sqrt{5}| \geq 0$. The special surd $\sqrt{5}$ must be used because then, as Götzky

Received 8 January 1958.

showed [2], each totally positive integer in k could be expressed as the sum of the squares of at most *four* integers in k. Later, Maass [3] made the more remarkable discovery that at most *three* squares would *always* suffice; in fact he arrived at a formula analogous to that of Gauss. Since Maass' formula is the basis of a machine calculation, we avoid irrelevant complexities by making certain further assumptions. First of all μ is to be free from square integral divisors in k except for powers of $(\sqrt{5}+3)/2 = [(\sqrt{5}+1)/2]^2 = \epsilon$. Secondly $a \geq 5b \geq 0$, since if $5b > a > \sqrt{5}b$, we can continually replace μ by μ/ϵ. Then $R(\mu)$, the number of representations of μ as the sum of three squares, is linked to an intrinsically positive quantity H, namely the class number of the bi-quadratic field generated by $\sqrt{5}$ and $\sqrt{-\mu}$, by means of the following formula (which excludes $\mu = 1$, $(5+\sqrt{5})/2$, and 3 where $H = 1$):

$$H = R(\mu)/G. \tag{2}$$

Here $G = 12$ when $(a, b) \not\equiv (1, 3)$, $(1, 5)$, $(2, 4)$, $(5, 1)$, $(5, 7)$, or $(6, 0)$ mod 8 (actually, when $\eta^2 + \mu \equiv 0 \bmod 4$ is unsolvable for η in k). Otherwise, $G = 120$ except when $(a, b) \equiv (1, 5)$, $(1, 11)$, $(5, 7)$, $(5, 9)$, $(6, 0)$, $(9, 3)$, $(9, 13)$, $(13, 1)$, $(13, 15)$, or $(14, 0)$ mod 16 (actually, when $\eta^2 + \mu \equiv 0 \bmod 8$ is solvable for η in k); in these cases, $G = 96$.

A tabulation of R and H for 446 selected values of $\mu = [a, b] = (a + b\sqrt{5})/2$ was made on the stored program electronic computer, the IBM 650. The values of a, b were selected with the restrictions

$$100 > a \geq 5b \geq 0,$$

and that μ have (except trivially for powers of ϵ) no square divisors in k; or, in terms of ordinary integral arithmetic, the condition is that d, the g.c.d. of $(\frac{1}{2}[a+b], \frac{1}{2}[a-b])$, be relatively prime to $5 (= [\sqrt{5}]^2)$, and both d and $(a^2 - 5b^2)/4d^2$ be square free. The machine assembled 446 such couples automatically into the highest four decimal positions of 446 individual ten digit storage locations, in lexicographic order.

The machine next tallied the decompositions $\mu = \xi_1^2 + \xi_2^2 + \xi_3^2$ where $\xi_i = [a_i, b_i]$, $a_i \equiv b_i \bmod 2$. Here the three couples ξ_i were scanned in lexicographic order, with the restriction $0 \leq b_i \leq 7$, while $a_i \geq 0$ and $a' < 100$, (see below). Thus, taking all sign possibilities, with

$$\begin{cases} a' = [a_1^2 + a_2^2 + a_3^2 + 5(b_1^2 + b_2^2 + b_3^2)]/2, \\ b' = \pm a_1 b_1 \pm a_2 b_2 \pm a_3 b_3, \end{cases}$$

the couples a', b' were constructed and compared with the 446 cases stored in the memory. Whenever a matching entry was located the count was augmented and accumulated in the last six decimal positions of the memory word. It might be appropriate to mention that the IBM 650 has a special "table look-up" operation that searches the memory at high speed for the appropriate entry. Without such an instruction the search would have had to be programmed with a considerable loss of running time.

In the final phase the "words" a, b, $R(\mu)$ were unpacked and the congruences were examined automatically to calculate H and to produce the output consisting of one IBM card per value of μ. (See attached table).

Tabulation of R and H for $\mu = (a + b\sqrt{5})/2$

a	b	R	H	a	b	R	H	a	b	R	H	a	b	R	H
2	0	6	1	33	3	240	2	47	9	144	12	59	1	144	12
4	0	12	1	33	5	288	3	48	2	216	18	59	3	168	14
5	1	24	1	34	0	96	8	48	4	168	14	59	5	96	8
6	0	32	1	34	2	96	8	48	6	144	12	59	7	96	8
7	1	24	2	34	4	120	1	49	1	144	12	59	9	144	12
9	1	24	2	34	6	72	6	49	3	240	2	59	11	144	12
10	2	24	2	35	1	96	8	49	5	288	3	60	2	144	12
11	1	24	2	35	3	144	12	49	7	96	8	60	6	192	16
12	0	48	4	35	7	48	4	49	9	120	10	60	8	192	16
12	2	48	4	36	2	96	8	50	2	144	12	61	1	288	3
13	1	96	1	36	6	96	8	50	6	144	12	61	3	144	12
14	0	96	1	37	1	240	2	50	8	96	8	61	5	144	12
14	2	24	2	37	3	168	14	51	1	144	12	61	7	240	2
15	1	48	4	37	5	96	8	51	3	96	8	61	9	360	3
15	3	48	4	37	7	288	3	51	5	168	14	61	11	96	8
16	2	24	2	38	0	384	4	51	7	144	12	62	0	576	6
17	1	48	4	38	2	96	8	51	9	96	8	62	2	240	20
17	3	120	1	38	4	96	8	52	0	144	12	62	4	216	18
18	2	72	6	38	6	144	12	52	2	144	12	62	6	264	22
19	1	48	4	39	1	72	6	52	6	192	16	62	8	480	5
19	3	48	4	39	3	96	8	52	8	120	10	62	10	192	16
20	2	48	4	39	5	72	6	52	10	192	16	62	12	240	20
21	1	120	1	39	7	96	8	53	1	360	3	63	1	288	24
21	3	48	4	40	2	96	8	53	3	168	14	63	3	192	16
22	0	192	2	40	4	96	8	53	5	144	12	63	7	240	20
22	2	72	6	40	6	144	12	53	7	480	5	64	2	120	10
22	4	72	6	41	1	72	6	53	9	480	5	64	4	168	14
23	1	72	6	41	3	288	3	54	2	120	10	64	6	144	12
24	2	72	6	41	5	120	1	54	4	168	14	64	10	96	8
24	4	48	4	41	7	96	8	54	6	96	8	64	12	168	14
25	1	48	4	42	0	192	16	54	8	384	4	65	1	144	12
25	3	192	2	42	2	168	14	54	10	144	12	65	3	480	4
26	0	96	8	42	4	360	3	55	1	144	12	65	7	192	16
26	2	48	4	42	6	144	12	55	3	192	16	65	9	144	12
26	4	120	1	43	1	120	10	55	7	144	12	65	11	384	4
27	1	120	10	43	3	192	16	55	9	144	12	65	13	240	2
27	3	96	8	43	5	144	12	55	11	96	8	66	0	192	16
27	5	96	8	43	7	144	12	56	2	144	12	66	2	144	12
28	0	96	8	44	0	144	12	56	4	96	8	66	4	360	3
28	2	96	8	44	2	96	8	56	6	120	10	66	6	144	12
29	1	192	2	44	6	96	8	56	10	120	10	66	8	144	12
29	3	48	4	44	8	72	6	57	1	216	18	66	10	192	16
29	5	72	6	45	1	384	4	57	3	576	6	66	12	240	2
30	2	96	8	45	3	96	8	57	5	480	4	67	1	240	20
30	4	96	8	45	7	240	2	57	7	192	16	67	3	192	16
30	6	48	4	46	0	288	3	57	9	192	16	67	5	216	18
31	1	48	4	46	2	120	10	57	11	360	3	67	7	168	14
31	3	72	6	46	4	72	6	58	0	288	24	67	9	312	26
31	5	48	4	46	8	288	3	58	2	144	12	67	11	216	18
32	2	72	6	47	1	96	8	58	4	360	3	67	13	240	20
32	4	120	10	47	3	216	18	58	6	288	24	68	0	288	24
32	6	120	10	47	5	192	16	58	8	192	16	68	2	192	16
33	1	144	12	47	7	120	10	58	10	120	10	68	6	288	24

Tabulation of R and H for $\mu = (a + b\sqrt{5})/2$—*Continued*

a	b	R	H	a	b	R	H	a	b	R	H	a	b	R	H
68	8	264	22	77	1	672	7	84	2	240	20	91	7	192	16
68	10	192	16	77	3	288	24	84	6	288	24	91	9	240	20
69	1	240	2	77	5	168	14	84	8	168	14	91	11	192	16
69	3	144	12	77	7	480	4	84	10	240	20	91	13	144	12
69	5	240	20	77	9	600	5	84	14	192	16	91	15	216	18
69	7	480	5	77	13	288	24	85	3	288	24	91	17	168	14
69	11	192	16	77	15	672	7	85	7	576	6	92	0	480	40
69	13	144	12	78	0	768	8	85	9	768	8	92	2	336	28
70	2	144	12	78	2	360	30	85	11	144	12	92	6	240	20
70	4	192	16	78	4	240	20	85	13	192	16	92	8	312	26
70	6	192	16	78	6	288	24	85	17	240	2	92	10	192	16
70	8	576	6	78	8	768	8	86	0	672	7	92	14	384	32
70	12	192	16	78	10	312	26	86	2	168	14	92	16	192	16
70	14	240	20	78	12	240	20	86	4	192	16	92	18	384	32
71	1	216	18	78	14	240	20	86	6	192	16	93	1	1056	11
71	3	192	16	79	1	192	16	86	8	384	4	93	3	384	32
71	5	120	10	79	3	264	22	86	10	144	12	93	5	360	30
71	7	144	12	79	5	144	12	86	12	264	22	93	7	720	6
71	9	144	12	79	7	168	14	86	14	144	12	93	9	480	4
71	11	168	14	79	9	168	14	86	16	384	4	93	11	408	34
71	13	144	12	79	11	144	12	87	1	384	32	93	13	288	24
72	2	360	30	79	13	144	12	87	3	288	24	93	15	768	8
72	4	336	28	79	15	144	12	87	5	360	30	93	17	864	9
72	6	192	16	80	2	192	16	87	7	432	36	94	0	480	5
72	10	216	18	80	4	192	16	87	9	288	24	94	2	240	20
72	12	192	16	80	6	336	28	87	11	264	22	94	4	240	20
72	14	216	18	80	12	144	12	87	15	336	28	94	6	264	22
73	3	672	7	80	14	240	20	87	17	264	22	94	8	480	5
73	5	600	5	81	1	144	12	88	2	264	22	94	10	144	12
73	7	192	16	81	3	480	4	88	4	192	16	94	12	216	18
73	9	288	24	81	5	480	5	88	6	336	28	94	14	216	18
73	11	360	3	81	7	216	18	88	10	264	22	94	16	480	5
73	13	576	6	81	11	672	7	88	12	336	28	94	18	240	20
74	0	192	16	81	13	360	3	88	14	288	24	95	1	288	24
74	2	144	12	81	15	192	16	89	1	168	14	95	3	240	20
74	4	240	2	82	0	384	32	89	3	576	6	95	7	288	24
74	6	168	14	82	2	216	18	89	5	360	3	95	9	240	20
74	8	144	12	82	4	480	4	89	7	192	16	95	11	192	16
74	10	144	12	82	6	336	28	89	9	240	20	95	13	240	20
74	12	480	4	82	8	192	16	89	11	480	4	95	17	240	20
74	14	144	12	82	10	312	26	89	13	576	6	95	19	192	16
75	1	192	16	82	12	720	6	89	15	168	14	96	4	216	18
75	3	192	16	82	14	240	20	89	17	168	14	96	6	240	20
75	7	288	24	82	16	240	20	90	2	240	20	96	10	312	26
75	9	192	16	83	1	240	20	90	4	480	4	96	12	240	20
75	11	240	20	83	3	288	24	90	6	288	24	96	14	240	20
75	13	192	16	83	5	240	20	90	8	288	24	96	18	192	16
76	0	144	12	83	7	288	24	90	12	480	4	97	1	288	24
76	2	144	12	83	9	360	30	90	14	336	28	97	3	600	5
76	6	192	16	83	11	168	14	90	16	288	24	97	5	768	8
76	8	192	16	83	13	288	24	91	1	168	14	97	7	336	28
76	10	144	12	83	15	336	28	91	3	240	20	97	9	456	38
76	14	144	12	84	0	192	16	91	5	192	16	97	11	960	10

Tabulation of R and H for $\mu = (a + b\sqrt{5})/2$—*Continued*

a	b	R	H	a	b	R	H	a	b	R	H	a	b	R	H
97	13	600	5	98	6	432	36	98	18	360	30	99	13	288	24
97	15	432	36	98	8	336	28	99	1	192	16	99	15	192	16
97	17	288	24	98	10	336	28	99	3	192	16	99	17	240	20
97	19	360	3	98	12	840	7	99	5	288	24	99	19	288	24
98	2	288	24	98	14	288	24	99	7	264	22				
98	4	840	7	98	16	240	20	99	11	192	16				

The computation was monitored for about the first 200 tally operations to make sure the score-keeping was correct in all possible cases. The tallying was, as before, basically a question of seeing that every permutation and change in sign in the triple (ξ_1, ξ_2, ξ_3) counted as a unit. The total running time was roughly 2.5 hours. One might remark that the human time involved in computing these class numbers H from basic *algebraic* concepts would have to be measured in "life-times," not "man-hours."

The computation was completed 18 April 1958 and was sponsored in part by the National Science Foundation Grant G-4222.

Department of Mathematics
University of Arizona
Tucson, Arizona

1. L. E. Dickson, *History of the Theory of Numbers*, v. III, G. E. Stechert, New York, 1934 (for references in the first paragraph).
2. F. Götzky, "Über eine zahlentheoretische anwendung von modulfunktionen zweier veränd-licher," *Mathematische Annalen*, v. 100, 1928, p. 411–437.
3. H. Maass, "Uber die darstellung total positiver zahlen des körpers $R(\sqrt{5})$ als summe von drei quadraten," *Abhandlungen aus dem Mathematischen Seminar der Hansischen Universitat*, v. 14, 1941, p. 185–192.

Multiplication Time on The IBM 709

By D. D. Wall

Average multiply time is useful for roughly estimating problem running time for various problems, as well as for roughly comparing different computing machines. Determining average multiply time for the 709 is complicated, however, due to its zero-skipping feature, and requires an investigation of runs of zeros in binary sequences. The particular problem we solve is that of evaluating $R(n, l)$ = total number of runs of length l in all the 2^n words of n bits each, and $S(n, l) = \sum_{x=l}^{n} R(n, x)$ = number of runs of length $\geq l$ in the 2^n words of n bits each. The resulting 709 average multiply time is 193 microseconds fixed point, or 170 microseconds normalized floating point, and the purpose of this note is to derive these two numbers.

We make use of a device which we call "differencing modulo 2," which obtains an $n - 1$ bit number from a given n bit number by writing 1 or 0 according as the successive bits in the given number exhibit a change or no change. For example, each of the (complementary) 8 bit numbers 11010001 and 00101110 gives the same result 0111001 as its 7 bit difference modulo 2.

Received 30 April 1958.

By differencing modulo 2, each run of length l in a given word of n bits gives rise to a run of 0's of length $l-1$ in the $n-1$ bit difference, and, conversely, each $n-1$ bit number with a run of $l-1$ 0's may be obtained by differencing either of two unique (complementary) n bit numbers with a run of length l. Finally, among all the 2^n numbers of n bits each, the number of runs of 0's of length l is the same as the number of runs of 1's of length l, and from these considerations we conclude that

$$R(n, l) = R(n-1, l-1) = \cdots = R(n-l+1, 1),$$

where, again, $R(n, l)$ = number of runs of length l in the 2^n words of n bits each.

We evaluate $R(n, 1)$ by counting the occurrences of an isolated 0 in each of the n possible positions of an n bit number: 2^{n-2} of these numbers begin 01, 2^{n-3} of these numbers begin 101, etc., and 2^{n-2} of these numbers end 10. Since the total number of runs is twice the number of runs of 0's, we have

$$R(n, 1) = 2[2^{n-2} + (n-2)2^{n-3} + 2^{n-2}] = (n+2)2^{n-2},\ n > 1, \text{ and } R(1, 1) = 2.$$

Therefore, $R(n, l) = (n-l+3)2^{n-l-1}$, $n > l$, and $R(n, n) = 2$.

To find the number of runs of length $\geq l$ we use the formula for summation by parts, $\sum_a^b u(x)\Delta v(x) = u(b+1)v(b+1) - u(a)v(a) - \sum_a^b v(x+1)\Delta u(x)$, and obtain

$$S(n, l) = \sum_{x=l}^{n} R(n, x) = (n-l+2)2^{n-l}.$$

Now, 709 multiplication requires 6μ for each 1 bit in the multiplier plus 6μ for each run of 0's, plus 6μ additional for each of these runs of 0's of length 7–12, plus 6μ more for each of these runs of length 13–18, etc., in addition to certain time charges independent of the bit structure. The average number of 1's in an n bit number, all n bit numbers equally likely, is $n/2$, and the average number of runs of 0's is $(n+1)/4$. This last quantity may be found by again differencing modulo 2, since the number of runs in an n bit number is 1 plus the number of 1's in the $n-1$ bit difference.

For fixed point multiplication, 12μ are required for instruction time, 12μ for data transfer, 6μ are reserved for other purposes, and 3μ must be added as an average waiting time at the end of the operation in order to total a multiple of 12μ basic cycles. Therefore, with $n = 35$ and all n bit numbers assumed equally likely, average fixed point multiply time is given by $T = 33 + 6(35/2) + 6(36/4) + d = 192 + d$, where $d = 6 \cdot \frac{1}{2} \cdot 2^{-35}[S(35, 7) + S(35, 13) + \cdots]$ is the delay occasioned by runs of 0's of length >6. On evaluating this expression we find that $d < 1$ and so obtain the 193μ result. Although the 709 disposes of a zero multiplier in a different and quicker manner, this exception does not modify the 193μ answer because of the assumption that all 2^{35} numbers are equally likely.

For normalized floating point multiplication, the mantissa has 27 bits of which the first is a 1, and 12μ are required to add the exponents. The resulting average time is $T = 45 + 6(1 + 26/2) + 6(27/4) + d = 169.5 + d$, and $d = 6 \cdot \frac{1}{2} \cdot 2^{-26}[S(26, 7) + S(26, 13) + \cdots]$ is $< \frac{1}{2}$, thereby giving the 170μ result.

International Business Machines Corporation
112 East Post Road
White Plains, New York

Diagonalization of Hermitian Matrices

By Carl-Erik Fröberg

As is well known, real symmetric matrices can be diagonalized very efficiently by the method of Jacobi, consisting of repeated two-dimensional rotations. A similar method can be constructed for Hermitian matrices (cf. ref. 1, p. 160); in this case the corresponding transformation matrices are unitary.

Apart from trivial factors the general two-dimensional unitary matrix has the form

$$U = \begin{pmatrix} \cos\varphi & -\sin\varphi\cdot e^{-i\theta} \\ \sin\varphi\cdot e^{i\theta} & \cos\varphi \end{pmatrix}$$

A two-dimensional Hermitian matrix $H = \begin{pmatrix} a & b-ic \\ b+ic & d \end{pmatrix}$ is transformed to diagonal form by $U^{-1}HU = D$, where θ and φ are defined by:

$$\begin{cases} \operatorname{tg}\theta = c/b \\ \operatorname{tg} 2\varphi = 2\sqrt{b^2+c^2}/(a-d) \end{cases}$$

The matrix elements of D are:

$$\begin{cases} d_{11} = \tfrac{1}{2}(a+d+R) \\ d_{22} = \tfrac{1}{2}(a+d-R) \\ d_{12} = d_{21} = 0 \end{cases} \qquad R^2 = (a-d)^2 + 4(b^2+c^2)$$

This procedure is now applied to Hermitian matrices of arbitrary order. Let

$$H = \begin{bmatrix} a_{11} & b_{12}-ic_{12} & b_{13}-ic_{13} & \cdots \\ b_{12}+ic_{12} & a_{22} & b_{23}-ic_{23} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

and define a real matrix $A = (\alpha_{ik})$ in the following way:

$$\begin{cases} \alpha_{ik} = b_{ik} & i < k \\ \alpha_{ii} = a_{ii} \\ \alpha_{ik} = c_{ki} & i > k \end{cases}$$

After a two-dimensional rotation (i, k) we arrive at a new Hermitian matrix H' and a corresponding real matrix $A' = (\alpha'_{rs})$. For computation of the elements α'_{rs} we introduce the following notations $(i < k)$:

$$\begin{cases} x_1 = \alpha_{ri}\cos\varphi \\ x_2 = \alpha_{ir}\cos\varphi \\ x_3 = \alpha_{rk}\cos\varphi \\ x_4 = \alpha_{kr}\cos\varphi \end{cases} \quad \begin{cases} y_1 = \alpha_{ri}\sin\varphi \\ y_2 = \alpha_{ir}\sin\varphi \\ y_3 = \alpha_{rk}\sin\varphi \\ y_4 = \alpha_{kr}\sin\varphi \end{cases} \quad \begin{cases} z_1 = y_3\cos\theta + y_4\sin\theta \\ z_2 = -y_3\sin\theta + y_4\cos\theta \\ z_3 = y_1\cos\theta - y_2\sin\theta \\ z_4 = y_1\sin\theta + y_2\cos\theta \end{cases}$$

Received 10 December 1957.

where

$$\begin{cases} \operatorname{tg} \theta = \alpha_{ki}/\alpha_{ik} \\ \operatorname{tg} 2\varphi = 2\sqrt{\alpha^2_{ik} + \alpha^2_{ki}}/(\alpha_{ii} - \alpha_{kk}) \end{cases}$$

Then we have for $r < i$ or $r > k$: for $i < r < k$:

$$\begin{cases} \alpha'_{ri} = x_1 + z_1 \\ \alpha'_{ir} = x_2 + z_2 \\ \alpha'_{rk} = x_3 - z_3 \\ \alpha'_{kr} = x_4 - z_4 \end{cases} \qquad \begin{cases} \alpha'_{ri} = x_1 - z_2 \\ \alpha'_{ir} = x_2 + z_1 \\ \alpha'_{rk} = x_3 - z_4 \\ \alpha'_{kr} = x_4 + z_3 \end{cases}$$

and further $\alpha'_{ii} = \frac{1}{2}(\alpha_{ii} + \alpha_{kk} + R)$, $\alpha'_{kk} = \frac{1}{2}(\alpha_{ii} + \alpha_{kk} - R)$, $\alpha'_{ik} = \alpha'_{ki} = 0$. Here $R^2 = (\alpha_{ii} - \alpha_{kk})^2 + 4(\alpha^2_{ik} + \alpha^2_{ki})$. When both r and s differ from i and k we have $\alpha'_{rs} = \alpha_{rs}$. The indices i and k can be chosen conveniently, e.g. to fulfill the condition $|\alpha_{ik}| + |\alpha_{ki}| = \max$.

The whole procedure is now repeated until all eigenvalues have been established with sufficient accuracy in the main diagonal.

This method has been successfully tested on SMIL, the electronic computer of Lund University, up to $n = 15$. The matrices had all elements in the main diagonal $= 1$, all upper-diagonal elements $= 1 - i$ and all lower-diagonal elements $= 1 + i$. The eigenvalues of these matrices can also be computed directly: $\lambda_k^{(n)} = \cot(\pi(4k + 1)/4n)$; $k = 0, 1, \cdots (n - 1)$. An accuracy of about 8 decimal digits was easily obtained; further it turned out that the time consumed was about 4 times longer than for the corresponding diagonalization of real symmetric matrices.

Department of Numerical Analysis
Sölvegatan 14
Lund, Sweden

1. A. S. HOUSEHOLDER, *Principles of Numerical Analysis*, McGraw-Hill Book Co., Inc., New York, 1953.
2. E. G. KOGBETLIANTZ, "Solution of linear equations by diagonalization of coefficients matrix," *Quart. Appl. Math.*, v. 13, 1955, p. 123–132.
3. J. GREENSTADT, "A method for finding roots of arbitrary matrices," *MTAC*, v. 9, 1955, 47–52.
4. M. LOTKIN, "Characteristic values of arbitrary matrices," *Quart. Appl. Math.*, v. 14, 1956, p. 267–275.

REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

86[C].—NBS Applied Mathematics Series, No. 53, *Table of Natural Logarithms for Arguments Between Five and Ten to Sixteen Decimal Places*, U. S. Government Printing Office, Washington, D. C., 1958, xiii + 506 p., 26 cm. Price $4.00.

This book tabulates $\log_e x$, $x = 5(.0001)10$, 16D, $\log_e x$, $x = 2(1)10$, 40D, and $\log_e(1 + x)$ and $-\log_e(1 - x)$, $x = 10^{-n}(10^{-n})10^{-n+1}$, $n = 1(1)13$, 25D. In addition there are sixteen constants given to 20D.

With AMS 31 (see *MTAC*, v. 8, Rev. 1167, 1954, p. 76) these give $\log_e x$ for $x = 0(.0001)10$, 16D and enough auxiliary material to make the computation of any natural logarithm to accuracy compatible with the tables reasonably con-

venient. (Actually, either volume suffices without intolerable inconveniences.) To this end there is a careful discussion of interpolation methods in an introduction written by Arnold N. Lowan. There are no tabulated differences or other aids to interpolation; linear interpolation yields 9D accuracy.

The volume is a reprint of one volume of the NYWPA "Table of Natural Logarithms," which appeared in four volumes. Two of the volumes (containing logarithms of the integers between 1 and 1 00000) evidently will not be reprinted, and the third is AMS 31 mentioned above.

Three trivial misprints and one worse misprint in the earlier printing are noted. The trivial misprints all involve arguments. The important misprint corrects the value of $\log_e 9.6061 = 2.26239\ 83133\ 487638$; the originally printed value was erroneous in the seventh through the twelfth significant digits.

The use of these valuable tables since their first printing in 1941 confirms their excellence and accuracy. The format is the same as that of AMS 31, which in this reviewer's eyes is perfectly adequate. The printing is easily legible, and the book is generally up to the high standards of this valuable series published by the National Bureau of Standards in scholarship and utility.

C. B. T.

87[D, E, L, S, X].—PHILIP M. MORSE & HERMAN FESHBACH, *Methods of Theoretical Physics*, Parts I & II, McGraw-Hill Book Company, Inc., New York, 1953. 23 cm., xxii × 1978 p. Price $30.00.

This remarkable two-volume work is already well known to most readers of *MTAC*, but a review seems to be in order as a means of indexing much of its content which relates to computation and numerical analysis.

The volumes are written by physicists for physicists, and they apply analysis to problems of physics without attempting to study the complete mathematical formulation or background of the methods used. However, the material presented certainly is as good a collection as is easily available of partial differential equations of mathematical physics. The treatment is largely by separation of variables and by transform methods.

There is no quarrel with the mathematical attainments of these authors, nor with their motivation (which, after all, is toward getting the answers to physical problems and hence entirely in keeping with the objectives of *MTAC*). The reviewer, however, has never fully accepted the development of the usual equation of the vibrating string through the use of infinitesimals, although it must be equivalent in the final analysis to a derivation using tension and inertial forces (the last integrated) over all portions of the string. Morse and Feshbach use the infinitesimal approach, whereas the integral approach is found, for example, in [1]. The reviewer also tends to view with some alarm the use of Dirac delta functions, introduced in Morse and Feshbach (p. 123) with a warning that "differentiation should be attempted only with considerable caution." This is still a minor criticism; the function gives valuable hints when properly used, the authors seem not to misuse it, and the argument here may be almost pedantic.

A more constructive approach here would be to turn to parts of the book which are likely to be of help to readers of *MTAC* in connection with numerical

analysis and computation. These include methods of solution, discussion of some of the more important special functions, and some tables.

Chapter 4 contains sections on gamma and elliptic functions, asymptotic series and the method of steepest descent, conformal mapping including the Schwarts-Christoffel transformation, and Fourier transforms (along with some mention of Laplace and Mellin transforms). There is a table of properties of and relations between gamma functions, elliptic functions and theta functions.

Chapter 5 contains a table of separable coordinates in three dimensions and a general table of second order differential equations and their solutions.

Chapter 6 contains a table of useful eigenfunctions and their properties (polynomials of Gegenbauer, Laguerre and Hermite).

Chapter 7 contains a short table of Green's functions.

Chapter 8 contains a section on applications of transforms to the solution of integral equations and a short table of integral equations and their solutions.

Chapter 9 is devoted to approximate methods—perturbation and variational methods. It includes a tabulation of methods.

Chapter 10, devoted to solutions of Laplace's and Poisson's equations, lists information about and expansions of trigonometric and hyperbolic functions, Bessel functions and Legendre functions.

Chapter 11, on wave equations, takes up cylindrical Bessel functions, Weber functions, Mathieu functions, spherical Bessel functions. It also includes a three page table of Laplace transforms.

Chapter 12, devoted to diffusion and wave mechanics, gets to Jacobi polynomials and semi-cylindrical functions.

Chapter 13 includes spherical vector harmonics.

Finally there is a set of tables. These include the following: for $x = 0(.2)8$, $\sin x$, $\cos x$, $\tanh x$, and e^{-x} all 4D, and $\tan x$, $\sinh x$, $\cosh x$, and e^{x} all 5S; for $x = 0(.05)2$, $\sin \pi x$, $\cos \pi x$, $\tanh \pi x$, and $e^{-\pi x}$, all 4D, and $\tan \pi x$, $\sinh \pi x$, $\cosh \pi x$, and $e^{\pi x}$ all 5S; the functions α, θ, χ and $|\zeta|$, 4D, and φ (in degrees) 2D, where $\tanh [\pi(\alpha - i\beta)] = \theta - i\chi = |\zeta| e^{(-i\varphi)}$, $\tanh \pi\alpha = 0(.05).95$, $\beta = 0(.05).45(.025).5$; the inverse of this function, giving α and β, 4D, for $\theta = 0(.2)4$ and $\chi = 0(.2)1.6(.4)2$; for $x = 0(.1)4(.2)12$, in χ, $\sinh^{-1} \chi$ and $\cosh^{-1} \chi$ (where real), 4D; spherical harmonics, $P_\alpha^\beta(x)$, $\alpha = 0(1)3$, $\beta = 0(1)\alpha$, $x = 0(.05)1$, 4D or 5S; Legendre functions, $P_\alpha^\beta(x)$ and $Q_\alpha^\beta(x)$, $\alpha = 0(1)2$, $\beta = 0(1)\alpha$, and $(\alpha,\beta) = (\pm\frac{1}{2}, 0)$, $(\pm\frac{1}{2}, 1)$, $(-\frac{1}{2}, 2)$, $(\frac{1}{2}, 0)$, $x = 1(.2)3(.5)8$, 4D or 5S, $P_\alpha^\beta(ix)$ and $Q_\alpha^\beta(ix)$, $x = 0(.2)2(.5)7.5$, 5D or 4 to 5S; Bessel functions $J_\alpha(x)$, $N_\alpha(x)$, $I_\alpha(x) = i^{-\alpha} J_\alpha(ix)$, $j_\alpha(x) = \sqrt{\pi/2x}\, J_{\alpha+\frac{1}{2}}(x)$ and $n_\alpha(x) = \sqrt{\pi/2x}\, N_{\alpha+\frac{1}{2}}(x)$, $x = 0(.1).2(.2)8$; (Here $N_\alpha(x)$ denotes a Bessel function of the second kind and is equivalent to $Y_\alpha(x)$ of Fletcher, Miller and Rosenhead [2]); Legendre functions $P_\alpha(\cos \nu)$, $\alpha = -1(1)9$, $\theta = 0(5)90°$, 4D; auxiliary functions $C_\alpha(x)$, $C_\alpha'(x)$, $\delta_\alpha(x)$ and $\delta_\alpha'(x)$ such that $J_\alpha(z) = C_\alpha(z) \sin [\delta_\alpha(z)]$, $dJ_\alpha(z)/dz = -C_\alpha'(z) \sin [\delta_\alpha'(z)]$, $N_\alpha(z) = -C_\alpha(z) \cos [\delta_\alpha(z)]$, $dN_\alpha(z)/dz = C_\alpha'(z) \cos [\delta_\alpha'(z)]$, δ and δ' in degrees, $\alpha = 0$, 1 and $x = 0(.1)1(.2)5$, $\alpha = 2$, 3 and $x = .1(.1).2(.2)5$, $\alpha = 4$, 5 and $\chi = .4(.2)5$, $\alpha = 6$, 7 and $\chi = 1(.2)5$, $\alpha = 8$, 9 and $x = 1.6(.2)5$, 4D or 5S and angles 2D; auxiliary functions $D_\alpha(x)$, $D_\alpha'(x)$, $\delta_\alpha(x)$ and $\delta_\alpha'(x)$ related analogously to $j_\alpha(x)$ and $n_\alpha(x)$, but this δ_α, δ_α' is different from that of the preceding table, $\alpha = 0$, 1 and $x = .1(.1)1(.2)5$, $\alpha = 2$, 3 and

$x = .1(.1).2(.2)5$, $\alpha = 4$, 5 and $x = .6(.2)5$, $\alpha = 6$, 7 and $x = 1(.2)5$, $\alpha = 8$, 9 and $x = 2(.2)5$, amplitudes 4D or 5S andangles (in degrees) 2D; periodic Mathieu functions, even and odd, $Se_\alpha(h, \cos x)$, $\alpha = 0(1)4$, and $So_\alpha(h, \cos x)$, $\alpha = 1(1)4$, $h^2 = 0(1)9$, $\chi = 0(10)90°$, 4D; quantities associated with periodic Mathieu functions, $M_\alpha^e(h) = \int_0^{2\pi} [Se_\alpha(h, \cos\chi)]^2 dx$, Je_α, C_α^e, δ_α^e such that $Je_\alpha + iNe_\alpha = -C_\alpha^e ie^{i\delta_\alpha^e}$ and analogous functions for odd functions, M_α^0, Jo_α', C_α^0, δ_α^0 with $Jo_\alpha' + No_\alpha' = iC_\alpha^0 e^{i\delta_\alpha^0}$, $h^2 = 0(1)9$, $\alpha = 1(1)4$ and $\alpha = 0$ for even functions, δ in degrees 2D, other quantities mostly 4D.

On the whole, this is an impressive collection of information, both in volume and in range of topics covered, and it represents an enormous effort on the part of its highly competent authors. References are adequate, although no attempt was made to include exhaustive bibliographies. Misprints detected by the rather cursory reading of the reviewer were trivial. Numerous and well chosen problems are included with each chapter. Reference is facilitated by summaries at the ends of the chapters, but the broad scope of the book precludes complete reference details in the fields covered. While the work is intended as a text book for a course the authors have presented at the Massachusetts Institute of Technology for several years, many people will find that it is more useful as a guide to be used in their courses rather than a text book to be followed faithfully; in this role of guide the book gives a valuable list of equations, solutions and significance. The nature of the information available in the book is described in the words of the authors as follows: "The present treatise, therefore, is primarily concerned with an exposition of the mathematical tools which have proved most useful in the study of the many field constructs in physics, together with a series of examples, showing how the tools are used to solve various physical problems. Only enough of the underlying physics is given to make the examples understandable.

"This is not to say that the work is a text on mathematics, however. The physicist, using mathematics as a tool, can also use his physical knowledge to supplement equations in a way in which pure mathematicians dare not (and should not) proceed. He can freely use the construct of the point charge, for example; the mathematician must struggle to clarify the analytic vagaries of the Dirac delta function. The physicist often starts with the solution of the partial differential equation already described and measured; the mathematician often must develop a very specialized network of theorems and lemmas to show exactly when a given equation has a unique solution. The derivations given in the present work will, we hope, be understandable and satisfactory to physicists and engineers, for whom the work is written; they will not often seem rigorous to the mathematician."

The Table of Contents follows:

Vol.	Chap.	
I	1.	Types of Fields
		1.1 Scalar Fields
		1.2 Vector Fields
		1.3 Curvilinear Coordinates
		1.4 The Differential Operator
		1.5 Vector and Tensor Formalism

1.6 Dyadics and Other Vector Operators
1.7 The Lorentz Transformation, Four-Vectors, Spinors
2. 2.1 The Flexible String
2.2 Waves in an Elastic Medium
2.3 Motion of Fluids
2.4 Diffusion and Other Percolative Fluid Motion
2.5 The Electromagnetic Field
2.6 Quantum Mechanics
3. Fields and the Variational Principle
3.1 The Variational Integral and the Euler Equations
3.2 Hamilton's Principle and Classical Dynamics
3.3 Scalar Fields
3.4 Vector Fields
4. Functions of a Complex Variable
4.1 Complex Numbers and Variables
4.2 Analytic Functions
4.3 Derivatives of Analytic Functions, Taylor and Laurent Series
4.4 Multivalues Functions
4.5 Calculus of Residues; Gamma and Elliptic Functions
4.6 Asymptotic Series: Method of Steepest Descent
4.7 Conformal Mapping
4.8 Fourier Integrals
5. Ordinary Differential Equations
5.1 Deparable Coordinates
5.2 General Properties, Series Solutions
5.3 Integral Representations
6. Boundary Conditions and Eigenfunctions
6.1 Types of Equations and of Boundary Conditions
6.2 Difference Equations and Boundary Conditions
6.3 Eigenfunctions and Their Use
7. Green's Functions
7.1 Source Points and Boundary Points
7.2 Green's Functions for Steady Waves
7.3 Green's Function for the Scalar Wave Equation
7.4 Green's Function for Diffusion
7.5 Green's Function in Abstract Vector Form
8. Integral Equations
8.1 Integral Equations of Physics, Their Classification
8.2 General Properties of Integral Equations
8.3 Solution of Fredholm Equations of the First Kind
8.4 Solution of Integral Equations of the Second Kind
8.5 Fourier Transforms and Integral Equations
II 9. Approximate Methods
9.1 Perturbation Methods
9.2 Boundary Perturbations
9.3 Perturbation Methods for Scattering and Diffraction
9.4 Variational Methods

10. Solutions of Laplace's and Poisson's Equations
    - 10.1 Solutions in Two Dimensions
    - 10.2 Complex Variables and the Two-dimensional Laplace Equation
    - 10.3 Solution for Three Dimensions
11. The Wave Equation
    - 11.1 Wave Motion on One Space Dimension
    - 11.2 Waves in Two Dimensions
    - 11.3 Waves in Three Space Dimensions
    - 11.4 Integral and Variational Techniques
12. Diffusion, Wave Mechanics
    - 12.1 Solutions of the Diffusion Equation
    - 12.2 Distribution Functions for Diffusion Problems
    - 12.3 Solutions of Schroedinger's Equation
13. Vector Fields
    - 13.1 Vector Boundary Conditions, Eigenfunctions and Green's Functions
    - 13.2 Static and Steady-state Solutions
    - 13.3 Vector Wave Solutions

C. B. T.

1. S. L. SOBALEV, *Uravneniia matematicheskoi fisiki* (*Equations of Mathematical Physics*), Moskva, 1950.
2. A. FLETCHER, J. C. P. MILLER, & L. ROSENHEAD, *An Index of Mathematical Tables*, Scientific Computing Service, 23 Bedford Square, London, W. C. 1, 1946. [See *MTAC*, v. 2, 1946–47, RMT 233, p. 13–18.]

88[F].—R. J. PORTER, *Irregular Negative Determinants of Exponent 3n* with their critical classes. Part III, from $-D = 1\ 00000$ to $1\ 50000$. 215 typewritten pages, 25.5 × 10 cm., deposited in *UMT* file.

This table of numbers occurring in the theory of quadratic forms completes the listing of "critical classes" to $-D = 1\ 50000$. The list of determinants is in three parts [*UMT* **155**, *MTAC*, v. 7, p. 34; *UMT* **185**, *MTAC*, v. 8, p. 96; Review **62**, *MTAC*, v. 9, p. 126]. The determinants are serially numbered and listed together with their critical classes in the present table [Review **84**, *MTAC*, v. 9, p. 126; Review 113, *MTAC*, v. 11, p. 275]. The lists were compiled with the aid of a *UMT* list of groups and series [Review **3**, *MTAC*, v. 9, p. 26] by the same author. The 58 values of $-D < 1\ 50000$ which have exponent of irregularity 9 are listed and discussed in [1]. The 14 values of $-D < 1\ 50000$ which have exponent of irregularity 6 are

17561	70244	91299	1 23539	1 28739
55555	80755	1 12723	1 25443	1 29355
67899	91083	1 19195	1 26755	

The other 5764 values of $-D < 1\ 50000$ have exponent of irregularity 3.

J. L. SELFRIDGE

University of California
Los Angeles, California

1. R. J. PORTER, "On irregular negative determinant of exponent 9*n*," *MTAC*, v. 10, 1956, p. 22–25.

89[F].—C. L. BAKER & F. J. GRUENBERGER, *Primes in the Thousandth Million*, 1958, 3 p. $8\frac{1}{2}'' \times 11'' + 39\ 17'' \times 22''$ blue line ozalid prints deposited in the UMT file.

A list of 47,957 primes found between, and including, 999,000,011 and 999,999,937 together with the first eighteen 10-digit primes.

This table came into being as a by-product of another table of primes: the 5,761,456 eight-digit primes, which will eventually be published.

A description of the method of construction of the tables on the IBM model 704 computer is included.

C. B. T.

90[F].—O. P. GUPTA, "Partitions into exactly k distinct primes," Panjab Univ., *Res. Bull.*, No. 107, p. 283–290, May 1957, Hoshiarpur, India.

The function tabulated is $R_k(n)$ the number of partitions of n into precisely k distinct primes > 1, for all possible values of k and for $n = 1(1)300$, together with the total number $R(n)$ of partitions of n into distinct primes. The function $R_k(n)$ was determined recursively by means of the auxiliary function $R_k(n, p_1)$ defined as the number of partitions of n into k distinct primes of which the smallest is p_1, so that

$$R_k(n, p_1) = \sum_{p > p_1} R_{k-1}(n - p_1, p)$$

and

$$R_k(n) = \sum_{p \geq 2} R_k(n, p).$$

Sample values are

$$R_2(300) = 21, \quad R_3(300) = 13504, \quad R_{15}(300) = 87, \quad R(300) = 53040.$$

D. H. LEHMER

Department of Mathematics
University of California
Berkeley, California

91[F].—M. S. CHEEMA & H. GUPTA, *Tables of Partitions of Gaussian Integers*. Natl. Inst. of Sciences, India, Mathematical Tables, v. 1, xii + 67 p., 1956. 30 × 24 cm. Price 15 Rps. Paper Bound.

The number $B(n, m)$ of partitions of the Gaussian integer $n + im$ into non-zero Gaussian integers of the first quadrant is tabulated for $m, n = 0(1)50$ in the first ten pages of the volume, even though $B(n, m) = B(m, n)$. In other words $B(n, m)$ is the number of partitions of the bipartite number (n, m) into bipartite summands (r, s) where $0 \leq r$, $0 \leq s$, $r + s > 0$. On p. 65 the number of unrestricted partitions

$$p(n) = B(n, 0)$$

is tabulated along with its sum function

$$p_1(n) = \sum_{k=0}^{n} p(k)$$

for $n = 0(1)50$. The rest of the volume is devoted to the auxiliary function $B_k(n, m)$ generated by

$$\left(\sum_{n,m=0}^{\infty} B_k(n, m)x^n y^m \right) \prod_{r=0}^{\infty} \prod_{t \geq k} (1 - x^r y^t) \prod_{s=1}^{\infty} (1 - x^s) = 1$$

which was used recursively to compute $B(n, m) = B_1(n, m)$. The printing is beautifully done.

D. H. LEHMER

Department of Mathematics
University of California
Berkeley, California

92[F].—HARVEY COHN, "Some bi-quadratic class numbers," [*MTAC*, this issue, p. 213–217.]

This is a tabulation of 446 sets (a, b, R, H) as is described in the attached note.

93[F, K].—RICHARD T. BURCH, *Approximate Values of Stirling Numbers of the Second Kind for the First Hundred Degrees*, Department of Defense, 134 p., multigraphed, 27 cm. Deposited in the UMT File.

The Stirling Numbers, $u(N, A)$, of the second kind are defined by the polynomial identity

$$x^N = \sum_{A=1}^{N} u(N, A)\, x^{(A)}, \quad \text{where} \quad x^{(A)} = \prod_{k=0}^{A-1} (x - k).$$

The recursion $u(N + 1, A) = AU(N, A) + u(N, A - 1)$ is used to calculate the numbers to four significant figures for N (the degree): $1(1)100$ and $A: 1(1)N$. A sum $U(N) = \sum_{k=1}^{N} U(N, k)$ and a sum check U' recursively computed from $U(N + 1) = 1 + \sum_{j=1}^{N} \binom{N}{j} U(j)$ are included. Values are expressed in 'floating' form: $1344 + 12$ standing for $1344 \cdot 10^{12}$. The number $u(N, A)$ is the number of (ordered) selections of N items from A categories, at least one item being selected from each category.

J. D. SWIFT

University of California
Los Angeles, California

94[G].—K. YAMAMOTO, "Structure Polynomial of Latin Rectangles and Its Application to a Combinatorial Problem," *Memoirs of the Faculty of Science*, Kyushu Univ., S. A., v. 10, No. 1, 13 p., 1956. Fukuoka, Japan.

The problem mentioned in the title is due to J. Touchard and asks for the number $N_n^{(k)}$ of permutations on

$$1, 2, 3, \cdots, n$$

which are discordant with the first k cyclic permutations

$$(1, 2, 3, \cdots, n)$$
$$(2, 3, 4, \cdots, 1)$$
$$\cdots\cdots\cdots\cdots$$
$$(k, k+1, \cdots, k-1)$$

This generalizes the famous problème des rencontres ($k = 1$) and the problème des ménages ($k = 2$). There is given (p. 13) a table of N_n in the case $k = 3$ for $n = 3(1)20$. Two auxiliary functions

$$N_n^* = V_n + 2, \quad N_n' = \{N_{n+1} + (-1)^n N^*_{n+1}\}/n$$

where

$$V_n = a^n + b^n \quad (a + b = 1, ab = -1)$$

in Lucas' companion to the Fibonacci series, are given for $n = 3(1)20$.

D. H. LEHMER

Department of Mathematics
University of California
Berkeley, California

95[G, P].—LORENZO LUNELLI, "Numerazione delle maglie in una rete completa," Instituto Lombardo di Scienze e Lettere, *Rendiconti*, v. 91, 1957, p. 903–911.

[G, P].—EMANUELE BIONDI, "Numerazione delle maglie in una rata qualsiasi," Instituto Lombardo di Scienze e Lettere, *Rendiconti*, v. 91, 1957, p. 912–926.

[X, Z].—LORENZO LUNELLI, "Determinazione delle maglie in une rete mediante una calcolatrice elettronica," Instituto Lombardo di Scienze e Lettere, *Rendiconti*, v. 91, 1957, p. 927–935.

These papers are concerned with the calculation of the number of polygonal subnets contained in networks of lines connecting nodes (as in electrical networks). In particular, the first paper derives a formula for the number of polygonal subnets contained in a "complete net" (wherein each pair of nodes is connected by one and only one line). The formula, for n nodes, is

$$M_c(n) = \sum_{q=3}^{n} \frac{(q-1)!}{2} \binom{n}{q}.$$

The derivation is straightforward.

The second paper derives in a somewhat different fashion a formula for the number of polygonal subnets contained in a *general* net. The procedure described considers the "complementary net" (the additional lines required to produce from the given net a complete net with the same number of nodes). Each set of ϕ lines contained in the complementary net is then classified into one of three types: 1) they form a closed polygon of ϕ lines. (In this case there is only one polygonal subnet which passes through all ϕ lines and the notation is used

$M_\phi^0(n) = 1$); 2) they form a number, γ, of separate connected sequences of lines (excluding those of the first case) such that at each node at most two lines meet. In these situations, the number of polygonal subnets passing through all ϕ lines is given by

$$M_\phi^\gamma(n) = \sum_{q=\gamma+\phi}^{n} 2^{\gamma-1}\binom{n-\gamma-\phi}{q-\gamma-\phi}(q-\phi-1)!, \quad \gamma+\phi \geqq 3;$$

3) all other situations (involving more than two lines meeting at some node). These must have zero polygonal subnets passing through all ϕ lines.

Let the complementary net of the given net consist of L lines. Let it have, for each ϕ, m_ϕ^γ sets of ϕ lines forming γ separate connected sequences. Finally, let $M_c(n)$ be the number of polygonal subnets in the complete net of n nodes. Then, the number of polygonal subnets in the given net, as derived in the paper, is given by the formula

$$M = M_c(n) + \sum_{j=1}^{L}(-1)^j \sum_{i=0}^{j} m_j^i M_j^i(n).$$

The paper then derives the formula for the number of polygonal subnets in a complete net by induction on n. Specifically, by the above formula,

$$M_c(n-1) = M_c(n) - (n-1)M_1^0(n) + \binom{n-1}{2} M_2^1(n).$$

It is then shown that $M_c(n) = M_c(n-1) + \frac{1}{2}(n-1)M_1^0(n)$. The formula for $M_c(n)$, as given in the first paper, is then derived by induction on n.

The third paper is concerned with the use of the CRC-102A at the Polytechnic Institute of Milan for determination of the number of polygonal subnets in a network. The procedure uses a representation of the set of lines in a given subnet as a vector with ones and zeros for its components—a one for a given component if the corresponding line in the original net is part of the subnet and a zero if the corresponding line is not part of the subnet. The original net itself is then represented by a matrix (the Poincaré matrix) with a row for each node showing by such a vector the subnet of lines meeting at the node. The columns of this matrix then indicate the endpoint nodes for each line. If the Poincaré matrix is multiplied onto the vector representing a given subnet, a product vector is obtained. The given subnet then is a polygonal subnet if and only if the product vector consists entirely of twos and zeros. The computer program examines in this manner each of the possible subnets of the given net and tabulates the polygonal subnets. The program includes an output conversion routine to represent the make-up of each polygonal subnet in convenient form for the human operator. The program is deliberately unsophisticated, and therefore uses no criteria other than the make-up of the product vector for determining the polygonal subnets.

R. M. HAYES

Business Systems Department
Magnavox Research Laboratory
Los Angeles, California

96[I, X, Z].—NATIONAL PHYSICAL LABORATORY, *Modern Computing Methods*, Notes on Applied Science No. 16, Published by National Physical Laboratory, Obtainable from Her Majesty's Stationery Office, London, England, 1957, vi + 128 p., 24 cm. Price 10s 6d. net.

This is a pamphlet which recounts the experience of professional computers attacking extensive problems on an automatic digital computer of considerable power. It is scholarly in that full use was made of literature known to these competent people, but it is not pedantic and it does not contribute particularly to the advanced research literature on numerical analysis.

However, the usefulness of this book to people who are meeting extensive computations for the first time is great, particularly because the larger and more detailed books in numerical analysis seem to be only mildly influenced by the development of large machines. The reviewer is consistent in recommending it to each person starting to work in his own group, and any careful reader of the pamphlet is likely to profit. So far as the reviewer knows the material in the book is not conveniently available in any other place, and considering the size of the pamphlet the choice of material presented is excellent.

Despite this approbation, it should be noted that this material is dated—many contributions to extensive computation have been made since the preparation of this material. Thus it should be augmented if it is to be used as a guide to a course in modern computation. Problems for students would also be required if it were to be used as a text. Numerous examples are worked out in the text, however, and these illustrate the methods and make reading easier. Occasionally the authors or compositors have been careless in displaying the results of computation. For example the publisher has noted that in Table 2 on page 49 a proper conclusion is drawn from incorrectly copied inputs to the table—a type of computing error which must be unusual among the distinguished group contributing to this book. On page 99 the right column of numbers in an example illustrating Euler's transformation is spaced artistically but not conveniently for following the example (which is easy to follow anyhow). Those of us who contend that time and effort spent on care and neatness in computing (and, from the reviewer's point of view, even more importantly in coding for automatic machines) are particularly good investments hope that these slight faults will be eliminated in later editions.

The table of contents follow:

Chapters

1. Linear Equations and Matrices (1)
2. Linear Equations and Matrices (2)
3. Roots of Polynomial Equations
4. Latent Roots of Matrices
5. Finite-difference Methods
6. Ordinary Differential Equations (1)
7. Ordinary Differential Equations (2)
8. Hyperbolic Partial Differential Equations
9. Elliptic and Parabolic Partial Differential Equations

10. Relaxation Methods
11. Tabulation of Mathematical Functions
12. Computation of Mathematical Functions

C. B. T.

97[K].—W. J. Dixon & F. J. Massey, Jr., *Introduction to Statistical Analysis*, McGraw-Hill Book Co., New York, 2nd ed., 1957, xii + 488 p., 24 cm. Price $6.00.

The first edition of this excellent textbook contained an unusually extensive and useful set of tables. An important feature of the second edition is the addition of 21 more tables. Besides this, 12 of the tables of the first edition have been extended in the second edition; five have been modified, one was reduced and one omitted. The present review will list the new tables and indicate the other changes. The new tables are:

A-4. Cumulative areas under the normal frequency curve to 4D for the standardized deviate = $-3.25(.05)3.25$.

A-8b (1) gives the .1, .5, 1, 2.5, 5, 10, 90, 95, 97.5, 99, 99.5, 99.9 percentiles of the standardized range to 2D for samples of N = 2(1)20 from normal.

A-8b (4). The variance and the efficiency to 3D or 3S of four estimates of the mean, the median, the midrange, the average of the best two ordered observations, and the average of all but two extreme observations in samples of N = 2(1)20 from normal.

A-8b (5). Standardized expected values of the order statistics to 3D in samples of N = 2(1)20 from normal.

A-8b (6). The best linear estimates of σ and their efficiencies to 3D for samples N = 2(1)10 from normal.

A-10b. Percentile values covering the range .005 to .125 to 3D for the number of occurrence in samples of N from the binomial with $p = \frac{1}{2}$ for $N = 3(1)100$.

A-12a. A chart showing the power curves for one- and two-sided tests for means and differences of means from normal with σ known for the significance levels .01 and .05 for two-sided tests and .005 and .025 for one-sided tests and for values of the test criterion in the range 0 to 4.5.

A-12b. For the same sampling situation as in A-12a, but with σ known or unknown, values of the test criterion to 2D or more required to make the power of the test .1(.1).9, .95, .975, .99, .995 for degrees of freedom 4(1)10, 12, 16, 24, 36, ∞ and for significance levels .005, .0125, .025, .05 for one-sided tests and .01, .025, .05, .1 for two-sided tests.

A-12c. For the same sampling situation as in A-12b, the sample sizes needed in the case of one mean and the equal sample sizes needed in the case of two means to attain the power .5(.1).9, .95, .99 for test criterion = .1, .2(.2)2, 3 for the significance levels, .005, .125, .025, .05 for one-sided tests.

A-19. Distribution of the signed rank statistic T. For samples of $N = 1(1)20$, the 100α and $100\ (1 - \alpha)$ percentile values of T for α's in the range .005–.125.

A-20. Distribution of the rank T'. For all pairs of sample sizes from (1, 1) to (10, 10), the cumulative probability distribution of T' is given to 3D.

A-28. Values of $p(1-p)$, $\sqrt{p(1-p)}$, $1-p^2$, $1-(1-p)^2$, 2 arcsin $\sqrt{p}$ and 2 arcsin $\sqrt{1-p}$ to 4 and 5D for $p = .01(.01)1$.

A-29a. Values of $\binom{N}{X} p^x(1-p)^{N-x}$ to 4D for $N = 2(1)10$ and $p = .01$, $.05(.05).5$ and $\frac{1}{3}$.

A-29b. All values of $\binom{N}{X}$ for $N = 1(1)20$.

A-29c. Values of $N!$ to 5S for $N = 1(1)100$.

A-30a. 95, 97.5, 99, 99.5, and 99.95 percentiles of the distribution of the coefficient of correlation to 3D in samples of $N = 5(1)20(2)30(10)60$, 80, 100, 250, 500, 1000, ∞ from an uncorrelated normal bivariate universe.

A-30b. Values of $\frac{1}{2} \ln \frac{1+r}{1-r}$ to 5D for $r = -.99(.01).99$.

A-30c. Selected values of Σd_i^2, used in the rank correlation coefficient, that will be equalled or exceeded with probabilities $\leq .175$ in samples of $N = 4(1)10$.

A-30d. The probabilities to 3D that the quadrant sum S, for the corner test for association, will be equalled or exceeded in samples of $2N$, for $N = 3(1)5$, 7, ∞ and $S = 8(1)23$.

A-31. Reciprocals of N to 5D for $N = 1(.01)1.1(.1)9.9$.

A-33. 4D values of $\sqrt{N}$ for $N = 1(.01)(10)(.1)99.9$.

Tables in the new edition which have been significantly extended beyond their counterparts in the first edition are:

A-5. Percentiles of the t distribution. 60, 70, 80 and 90 percentile values have been added and 99.75 percentile values deleted.

A-6b. Percentiles of the χ^2/df distributions. .05, .1, 20(10)80, 99.9 and 99.95 percentile values have been added and the table now gives values for 1(1)20(2)30(5)60(10)100(20)200(50)500, 750, 1000, 5000 and ∞ d.f.

A-7c. This is a much extended table of percentiles of the F distribution. For the degrees of freedom, $\nu_1 = 1(1)12$, 15, 20, 24, 30(10)60, 100, 120, 200, 500, ∞ and $\nu_2 = 1(1)12$, 15, 20, 24, 30, 40, 60, 120, ∞, the values of F to 3S or 3 or more D are given for which the cumulative probabilities are .0005, .001, .005, .01, .025, .05, .1, .25, .5, .75, .9, .95, .975, .99, .995, .999, and .9995.

A-8b(1). This now gives for samples of 2(1)20 from normal, unbiased estimates of σ as a multiple of the sample range to 3D, the variance of the estimate to 3S and its efficiency to 3D.

A-8b(3). Modified linear estimates of σ in samples of N from normal are now given for $N = 2(1)20$ with the coefficients to 4D together with their variances to 3S and the efficiencies to 3D.

A-8c(1). Upper (and lower) 95, 97.5, 99, 99.5, 99.9 and 99.95 percentiles of the ratio of the deviation of the sample mean from the population to the range in samples of N from normal for $N = 2(1)20$ are given to 3D for the first four percentiles and to 2D for the last two.

A-8c(2). In the new tables the same set of percentiles is given as in A-8c(1) and the same D for the difference of two sample means divided by the average of the ranges for sample of sizes $N_1 = N_2 = 2(1)20$.

A-8d. For the ratio of the ranges in samples of N_1 and N_2 from normal (substitute F ratio), .5, 1, 2.5, 5, 95, 97.5, 99 and 99.5 percentiles are now given to 2 or 3S again for N_1, $N_2 = 2(1)10$.

A-8e. Criteria for testing for extreme mean. Table 8e of the first edition is extended to give critical values at the .5%, 2%, 10%, 20% and 30% levels as well as the 1% and 5% levels.

A-11. This differs in form from Table 11 of the first edition. It now gives the complete distribution to 3D of the total number of runs in samples of (N_1, N_2) for $N_1, N_2 = 2(1)10$. For $N_1 = N_2 = 11(1)20(5)100$, the table gives the number of runs that occur with probabilities $\leqq .005, .01, .025, .05, .95, .975, .99, .995$, the mean number of runs to the nearest integer, and the variance and the standard deviation to 2D.

A-13. Table 13 of the first edition is replaced by charts for the power of the analysis of variance test. If for K samples of n each from $N(\mu_i, \sigma^2)$; $i = 1, \cdots, K$; we designate by φ^2 the average squared deviation of the μ_i's from their mean divided by σ^2/n, the charts enable one to read the power of the test with significance levels .01 and .05 for the appropriate range of values of φ for the degrees of freedom, $n = 1(1)8$ and $n_2 = 6(1)10, 12, 15, 20, 30, 60, \infty$.

A-14. The values of $\ln (1 - \beta)/\alpha$ to 3D are now given for $\alpha, \beta = .001, .005, .01, (.01).05(.05).25$.

Table 15. The values of the cumulative Poisson distribution to 3D are now given for $Np = .05(.05)1(.1)2(.2)4$.

A-16. Tolerance factors for normal distributions. This is the same as table 16 of the first edition with the exception that the tabulation is for sample sizes, $N = 2(1)27$, $30(5)100(10)200$, 250, $300(100)1000$, instead of for $N = 2(1)102(2)180(5)300(10)400(25)750(50)1000, \infty$.

A-18. Percentiles of the distribution of the ratio of the range to the standard deviation (as defined in this text) in samples of K from normal. Upper 95 and 99 percentiles are given for $\nu = 1(1)9$ degrees of freedom and upper and lower values for $\nu = 10(1)20, 24, 30, 40, 60, 120, \infty$, all to 3S for $K = 2(1)20$.

Table 19. Determination of the second sample size, of the first edition is omitted from the second edition.

This extensive and useful set of tables in itself very considerably enhances the value of the second edition as compared to the first.

C. C. C.

98[K].—B. L. VAN DER WAERDEN, *Mathematische Statistik* Springer-Verlag, Berlin, Germany, 1957, ix + 360 p., 23 cm. Price DM 49.60.

This is an example of a book written in the careful style of van der Waerden. It includes at the beginning a chart of the type used in his famous *Algebra* indicate earlier chapters which are essential to an understanding of later ones. At the end it includes an index of examples classified according to the field in which they were generated, and there is a short English-German glossary.

The purpose of this review is to announce tables in the book, and not particularly to discuss material covered. However, it might be noted that the author describes the central question of statistics as that of how far quantities estimated from random samples may vary from the corresponding ideal values (translation approximate only). To attack this question he studies the axiomatic development of probability and its applications to statistics, including studies of confidence limits and confidence bands. He then turns to the theory of estimation, and then to the testing of hypotheses.

Tables include the normal distribution (area) $\Phi(t)$, $t = -3.9(.1)-3(.01)-3(.1)3.9$, 4D, the inverse function for the normal distribution $\Psi(x)$, $x = 0(.001).99$, 2D; g and g^2, the (one sided) .05, .1, .5, 1, 2.5 and 5% points for the normal distribution and the χ^2 distribution with one degree of freedom, 2D; exact and asymptotic 1 and 5% points for Smirnoff's test, $n = 5, 8, 10(10)50$, 4D and the ratio of the values 3D; exact and asymptotic 1 and 5% points for Kolmogorov's test for $n = 5(5)30(10)80$ and the 5% values for $n = 90, 100$, 4D with ratios 3D; .1, 1, and 5% points of χ^2 with f degrees of freedom, $f = 1(1)100$, 3S with suitable formula for larger f; (one sided) .05, .5, 1, and 2.5% points for Student's test with f degrees of freedom, $f = 1(1)30(10)60(20)100, 200, 500, \infty$, 4S; 5% points for F-test with f_1 and f_2 degrees of freedom in numerator and denominator respectively, $f_1 = 1(1)20(2)30(10)60(20)100$, $f_2 = 1(1)30(2)50(10)100(25)150, 200, 300, 500, 1000$, 3D; the 1% points with f_1 as above and $f_2 = 2(1)30(2)50(5)60-(10)100(25)150, 200, 300, 500, 1000$, and the .1% points for $f_1 = 1(1)10(5)20, 30, 50, 100$ and $f_2 = 2(1)20(2)30(10)60(20)100, 200, 500, \infty$, 3S; (one sided) .5, 1 and 2.5% points for the sign test, $n = 5(1)100$, integral; tables for Wilcoxon's test of the number of inversions in a series with g marks x and h marks y with $g = 2$ and $h = 5(1)10$, and $g = 3(1)10$, $h = g(1)10$, the probabilities not exceeding 5% are listed 3S; tables for the X-test, (one sided) .5, 1, and 2.5% points for $X = \sum_{r \in z} \left(\frac{r}{n+1} \right)$, and corresponding values of X for $n = 6(1)50$ and $g - h = 0(1)5$, 3S, where g and h are as in Wilcoxon's test above, $n = g + h$, r is the set of rank numbers of x in the sequence of x and y, and ψ is the inverse normal distribution function; also an auxiliary table of $Q = \frac{1}{n} \sum_{r=1}^{n} \psi^2 \left(\frac{r}{n+1} \right)$, $n = 1(1)150$, 3D; (one sided) .1, 1, 2, and 5% points for the correlation coefficient, $f = n - 2 = 1(1)20(5)50(10)100$, 3D. Many of these are reprinted from other sources, which are acknowledged.

C. B. T.

99[K].—H. A. DAVID, "The ranking of variances in normal populations," Amer. Stat. Assn., *Jn.*, v. 51, 1956, p. 621–626.

This paper considers two procedures for ranking, or grouping, the variances σ_i^2 $(i = 1, \cdots, k)$ of k normal populations by use of mean square estimates s_i^2 each based on ν degrees of freedom.

Let $X_1 < X_2 < \cdots < X_k$ be an ordered sample of k independent observations from a unit normal distribution. Table I gives, to 3S and for $k = 2(1)12$, values G such that the expected number of differences, $X_{i+1} - X_i$, which exceed G is $\alpha = .05$ and $\alpha = .01$. The title of Table I refers to these as "Critical $100\alpha\%$ levels . . ." and the text material points out that they are not percentage points. If $X_i = \log s_i^2$ Table I can be used to rank the variances σ_i^2.

Table II gives, to 3S, for $k = 2(1)12$, for $\nu = 2, 4(1)10, 12, 15, 20, 30, 60, \infty$, and for $\alpha = .05$, $\alpha = .01$, values of $R\alpha(\nu)$ such that the expected number of ratios s^2_{i+1}/s_i^2 (when the s_i^2 have been ordered by size) which exceed $R\alpha(\nu)$ is α. Again the title refers to "Critical $100\alpha\%$ levels. . . ." The title has a misprint, the division bar being printed as m.

Table III (titled "Significant Maximum F-Ratios in a 1% Level Multiple F_{max}-Ratio Test. (Normal Variation Assumed)") gives, to 3S, for $\alpha = .01$, for $k = 2(1)12$, and for $\nu = 2(1)10, 12, 15, 20, 30, 60, \infty$, values $C\alpha(\nu, k)$ which are the upper $100\{1 - (1 - \alpha)^{k-1}\}\%$ points of the maximum ratio of the s_i^2.

F. J. MASSEY, JR.

University of California
Los Angeles 24, California

100[K].—D. G. CHAPMAN, "Estimating the parameters of a truncated gamma distribution," *Ann. Math. Stat.*, v. 27, 1956, p. 498–506.

To facilitate computation of maximum likelihood estimates of parameters of the gamma or Type III distribution with density function

$$f(x) = \frac{a^b}{\Gamma(b)} e^{-a(x-c)}(x - c)^{b-1}; \quad x \geq c,$$
$$= 0; \qquad\qquad x < c,$$

in the case $c = 0$, a table of the function

$$b = \gamma^{-1}[\ln \bar{x} - \overline{\ln x}],$$

where $\bar{x}$ designates the sample mean and $\overline{\ln x}$ designates the mean of natural logarithms of the sample observations, is given for $[\ln \bar{x} - \overline{\ln x}] = .01(.001).99$, $.1(.005).495$, $.50(.01)1$. For values of the argument from $.01 - .049$, b is given to 4D. Elsewhere it is given to 3S. The author states that a complete tabulation of $\gamma(b)$ with its first and second differences, correct to one figure in the fifth decimal, for $b = .01(.01)2$, $2(.02)5$, $5(.1)20$, $20(1)100$, is available in mimeographed form from the Laboratory of Statistical Research, University of Washington.

A new procedure bearing certain resemblances to a minimum chi square method is suggested for estimating parameters of a truncated gamma distribution. The asymptotic variance-covariance matrix of these estimates is determined.

The author points out that his method is applicable to a number of other truncated distributions whether truncation is in the tails or the center of the distribution.

A. C. COHEN, JR.

The University of Georgia
Athens, Georgia

101[K].—R. J. BUEHLER, "Confidence intervals for the product of two binomial parameters," Amer. Stat. Assn., *Jn.*, v. 52, 1957, p. 482–493.

The author considers the problem in which two binomial populations have been sampled and that k_1 failures are observed in a sample of n_1 from the first population and k_2 failures are observed in a sample of n_2 from the second and one is interested in finding the number $C_{n_1 n_2}(k_1, k_2; \alpha)$ having the property that

$$\text{Prob}\{0 \leq P_1 P_2 \leq C_{n_1 n_2}(k_1, k_2; \alpha)\} \geq \alpha \quad \text{for all} \quad 0 \leq P_1, P_2 \leq 1$$

where P_1 and P_2 are the probabilities associated with the respective statistically independent populations. That is, one is interested in obtaining an upper confidence interval in the sense of Neyman for the product of two probabilities of statistically independent binomial populations.

The author proposes that the above problem be solved by taking

$$C_{n_1 n_2}(k_1, k_2; \alpha) = C_{n_1}(k_1, \sqrt{\alpha}) \cdot C_{n_2}(k_2, \sqrt{\alpha})$$

where $C_{n_1}(k_1, \sqrt{\alpha})$ and $C_{n_2}(k_2, \sqrt{\alpha})$ are such that

$$\text{Prob}\{0 \leq P_i \leq C_{n_i}(k_i; \sqrt{\alpha})\} \geq \sqrt{\alpha} \quad \text{for} \quad 0 \leq P_i \leq 1, i = 1, 2.$$

Approximate solution to the above inequalities are obtained under the assumption that n is large and C is small by using Poisson terms for the binomial terms. These approximations are tabulated.

Table 1 gives values of $nC_n(k; \alpha)$ to 3S for $k = 0(1)6, 8, 10, 20, 30$ and $\alpha = .90, .95$ and $.99$.

To obtain the confidence limit $C_{n_1 n_2}(k_1, k_2; \alpha)$ which are tabulated in Table 3, the author resolves the lack of uniqueness problem associated with the solution of the inequality by imposing a generalized condition of (1) Regularity: If $k_1 < k_2$ then $C_n(k_1; \alpha) < C_n(k_2, \alpha)$ and (2) Shortness: $C_n(k; \alpha)$ should be as small as possible. The 3S values for $n_1 n_2 C_{n_1 n_2}(k_1, k_2; \alpha)$ are tabulated for $k_1 = 0(1)5$ and $k_2 = 0(1)29$ for $\alpha = .90$ and $.95$. The symmetry allows the interchange of k_1 and k_2.

The final Table 5 gives unsymmetrical values for the confidence limits $n_1 n_2 C_{n_1 n_2}(k_1, k_2; 0.9)$ to 1D for $k_1, k_2 = 0(1)4$ which are shortened by use of an auxiliary random element in a fashion similar to that suggested by Stevens [1] and Eudey [2].

C. F. KOSSACK

Purdue University
Lafayette, Indiana

1. W. L. STEVENS, "Fiducial limits of the parameter of a discontinuous distribution," *Biometrika*, v. 37, 1950, p. 117–129.
2. M. W. EUDEY, "On the treatment of discontinuous random variables," *Stat. Lab. Technical Report* No. 13, Univ. of Calif., Berkeley, 1949.

102[K].—JOSEPH BERKSON, "Tables for the maximum likelihood estimation of the logistic function," *Biometrics*, v. 13, 1957, p. 28–34.

Table 2 in this paper gives to 5D the function $p = 1/[1 + e^{-l}]$ and (directly beneath the value of p) the function $w = p(1 - p)$, for argument $l = 0(.01)4.99$. The text states that the tables are correct to ± 1 in the fifth place and that linear interpolation in the table will also give values correct to ± 1 in the fifth place.

Suppose $P(x) = 1/[1 + e^{-(\alpha+\beta x)}]$. Then $P(x)$ is said to be a logistic function of x. If, for each of a set of values $x_1, \cdots, x_k$, there are performed n_i independent binomial trials with probability of success $P_i = P(x_i)$, the results may be used to estimate the parameters α and β in a number of ways. The table of values of the functions p and w facilitates the calculation of the Maximum Likelihood estimates of α and β. The text explains the method and gives a numerical example (table 1).

PAUL MEIER

University of Chicago
Chicago, Illinois

103[K].—A. J. DUNCAN, "Charts of the 10 per cent and 50 per cent points of the operating characteristic curves for fixed effects analysis of variance F tests, $\alpha = 0.01$ and 0.05," Amer. Stat. Assn., *Jn.*, v. 52, 1957, p. 345–349.

The symbols and expressions α, β (ordinate of OC curve), f_1, f_2, OC(1-power) have meanings usual in testing hypotheses and (fixed effect) variance analysis. Duncan gives four (very small) charts for ϕ (defined below)

(1) $\alpha = .05, \beta = .10, f_1 = 1(1)8, f_2 = 6(1)10, 12, 15, 20, 30, 40, 60, \infty$

(2) $\alpha = .05, \beta = .50, f_1$ as above, f_2 as above

(3) $\alpha = .01, \beta = .10, f_1$ as above, f_2 as above

(4) $\alpha = .01, \beta = .50, f_1$ as above, f_2 as above

where Φ is defined for (a) one-way analysis, (b) two-way analysis, (c) Latin Squares. For example, for a Latin Square,

$$\phi^2 = \sum_k \delta_k^2/\sigma^2$$

for testing treatment effects, the basic linear model being

$$X_{ij} = \mu + \tau_i + \theta_j + \delta_k + \epsilon_{ijk}.$$

These charts (described by the author as "essentially a special condensation of the Pearson and Hartley charts") should be examined along with similar (power) tables and charts by Fox, [1], Lehmer [2], Pearson and Hartley [3], and the pathbreaking paper by Tang [4].

HAROLD FREEMAN

Massachusetts Institute of Technology
Cambridge, Massachusetts

1. MARTIN FOX, "Charts of the power of the F-test," *Ann. Math. Stat.*, v. 27, 1956, p. 484–497. [*MTAC*, v. 11, Rev. 28.]
2. EMMA LEHMER, "Inverse tables of probabilities of errors of the second kind," *Ann. Math. Stat.*, v. 15, 1944, p. 388–398.
3. E. S. PEARSON & H. O. HARTLEY, "Charts of the power function for analysis of variance tests derived from the noncentral F-distribution," *Biometrika*, v. 38, 1951, p. 112–130. [*MTAC*, v. 8, 1954, *RMT* 1183, p. 84.]
4. P. C. TANG, "The power function of the analysis of variance test with tables and illustrations of their use," *Stat. Res. Memoirs*, v. 2, 1938, p. 126–149 + tables.

104[K].—BRUNO DE FINETTI, "Una legge riguardante l'estinzione nei processi di eliminazione," *Giornale dell' Instituto Italiano degli Attuari*, v. 15, 1953, p. 94–99.

A population, initially of n elements, is subjected to successive trials with elimination probability q (for each individual, each trial) until complete extinction. The author gives a recurrence formula and an exact series formula for P_m^n = the probability that the number of survivors eliminated at the last step (before extinction) is m, $m = 1, 2, \cdots, n$. An approximate formula is derived, $P_m = q^m/m|\log(1-q)|$, which, [1], is not the limit of P_m^n as n becomes infinite. As is shown in [1], P_m^n approaches a periodic function of $\log n$ for whose values P_m is an average for large n.

A table of P_m^n is given for the cases through $n = 10$ for $q = \frac{1}{2}$. Exact expressions are indicated and values (quotients) are given up to maximum 6S. There is an error of +1 in the least significant digit for P_6^6. Round-off is uniform over n for fixed m but (starting at $n = 3$) works to the disadvantage of $\sum_m P_m^n = 1$. It is of interest to note that for the case of $q = \frac{1}{2}$: $P_3^3 = P_2^3$; $P_9^{10} < P_8^{10} < P_7^{10} < P_{10}^{10}$.

W. H. MARLOW

The George Washington University
Logistics Research Project
Washington, D. C.

1. GIUSEPPE OTTAVIANI, "A proposito della legge di estinzione nei processi di eliminazione," *Giornale dell' Instituto Italiano degli Attuari*, v. 15, 1953, p. 100–114.

105[K, W, Z].—ALISON DOIG, "A bibliography on the theory of queues," *Biometrika*, v. 44, 1957, p. 490–514.

An extensive and valuable bibliography on the theory of queues including related topics and papers devoted to computational aspects of the subject.

C. B. T.

106[L].—K. A. KARPOV & S. N. RAZUMOVSKI, *Tablitsy integral'nogo logarifma* (*Tables of the logarithmic integral*). Akad. Nauk SSSR, Vychislitel'ny Tsentr, *Matematicheskie Tablitsy* (Acad. Sci. USSR, Computational Center, *Mathematical Tables*). Izdatel'stro Akad. Nauk SSSR (Press of the Academy of Sciences of the USSR). Moscow, 1956. 320 p., 26.7 cm. Price 33.30 rubles.

The main table (pages 11–311) lists values of the logarithmic integral

$$\operatorname{li} x = \int_0^x dt/\ln t \quad (x \geq 0),$$

understood as a principal value when $x > 1$, to 7S for

$$x = 0(.0001)2.5(.001)20(.01)200(.1)500(1)10{,}000(10)24{,}990.$$

There are no differences except in the first five pages ($x \leq .5$), where first and second differences are provided.

Pages 312–315 contain an auxiliary table of $\operatorname{li} x - \ln|1 - x|$ to 6D without differences for $x = .95(.0001)1.0499$. Page 316 tabulates $\frac{1}{2}t(1 - i)$ to 4D without differences for $t = 0(.001)1$.

The main table was computed on the high-speed electronic calculating machine (BÉSM) of the Academy of Sciences. It was found expedient to use, chiefly, numerical quadrature by Simpson's rule, reserving the formula $\operatorname{li} x = \operatorname{Ei}(\ln x)$ for the computation of pivotal and checking values. The error is stated not to exceed 0.6 final units. A correction slip pasted in gives the following correction on p. 186:

$$\operatorname{li} 25.82, \quad \textit{for} \quad 0.1175643 \cdot 10^2 \quad \textit{read} \quad 0.1176643 \cdot 10^2.$$

Although the range $0 \leq x \leq 24{,}990$ does not include the high values (up to at least $x = 10^8$) of interest in number theory, and although extra figures may be found elsewhere for a few arguments, this is nevertheless by far the largest table of the logarithmic integral available. It will be noticed that throughout the interval $.0100 \leq x \leq 24{,}990$ there is always at least a three-figure argument, and that the main table runs to 300 pages. None of its predecessors comprises more than a page or two. In these circumstances one cannot seriously dissent from the claim that this is the first table of the logarithmic integral which is anything like complete. It can hardly fail to be found of use.

On pages 318–319 are given corrections to four other tables published in the same series in 1954 and 1955.

A. F.

107[L].—E. N. Dekanosidze, *Tablit͡sy t͡silindri͡cheskikh funkt͡siĭ ot dvukh peremennykh* (*Tables of cylinder functions of two variables*). Akad. Nauk SSSR, Vychislitel'ny t͡sentr, *Matematit͡cheskie Tablit͡sy* (Acad. Sci. USSR, Computational Center, *Mathematical Tables*). Izdatel'stvo Akad. Nauk SSSR (Press of the Academy of Sciences of the USSR), Moscow, 1956. 494 p. + 2 leaves of diagrams, 26.7 cms. Price 50 rubles.

These tables were computed at the Institute of Precise Mechanics and Computational Technique on the initiative of Professor P. I. Kuznet͡sov. The functions tabulated are Lommel functions of two variables, $U_\nu(w, z)$, $V_\nu(w, z)$, which are related to Bessel functions and were first introduced in connection with the study of diffraction. Apart from the use of ν and w instead of n and y, the functions are as described in *MTAC*, v. 1, 1944, p. 258 and FMR *Index*, p. 301.

The tables (pages 11–492) are for $\nu = 1$ and 2, the four functions U_1, U_2, V_1, V_2 being listed in parallel columns. All values are to 6D and there are no differences. There is a separate table for each of the following values of w:

$$w = .5(.02)1.2(.05)4(.1)10.$$

For $w \leq 6.2$, the second variable takes the values of $z = w(.01)W$, where W is always fairly close to $4\sqrt{w}$; for $w \geq 6.3$, on the other hand, $z = w(.01)10$. The resulting region of tabulation in the (w, z) plane is illustrated in Figure 3.

Various formulas are given for extending the scope of the tables. For example, computation for suffixes ν which are integers greater than 2 is possible if the present tables are supplemented by tables of Bessel functions J of integral order; negative values of w and z are covered by symmetries; while a further valuable resource is that, for integral ν, the U and V functions are interchanged if w is replaced by z^2/w. Asymptotic expansions for large w and z are given. Some numerical examples are worked out.

The computation of the tables is briefly sketched in mathematical terms, but nothing is said about the machine or machines used. The errors are stated not to exceed unity in the sixth decinal place. The mere four pages of the Introduction contain so many formulas that they will be largely intelligible to mathematicians without a knowledge of Russian.

A. F.

108[L, S].—KURT ALDER & AAGE WINTHER, *Tables of the Classical Orbital Integral in Coulomb Excitation*, Mat. Fys. Medd. Dan. Vid. Selsk., Bind 31, No. 1, 1956, 74 p., 9 cm. Price 3.00 kr.

This paper contains a tabulation of certain functions which arise in the classical description of Coulomb excitation. The functions tabulated here are the orbital integrals $I_{\lambda\mu}(\nu, \zeta)$, defined by

$$I_{\lambda\mu}(\nu, \zeta) = \int_{-\infty}^{+\infty} \{e^{i\nu[\epsilon \sinh w + w]}\} \frac{[\cosh w + \epsilon + i\sqrt{\epsilon^2 - 1} \sinh w]^\mu dw}{[\epsilon \cosh w + 1]^{\lambda+\mu}},$$

the differential cross-section functions $\frac{df_{E\lambda}(\nu, \zeta)}{d\Omega}$, and $\frac{df_{M\lambda}(\nu, \zeta)}{d\Omega}$ given by

$$\frac{df_{E\lambda}(\nu, \zeta)}{d\Omega} = \frac{4\pi^2}{(2\lambda + 1)^3} \sum_\mu \left| Y_{\lambda\mu}\left(\frac{\pi}{2}, 0\right)\right|^2 |I_{\lambda\mu}(\nu, \zeta)|^2 \sin^{-4} \nu/2,$$

$$\frac{df_{M\lambda}(\nu, \zeta)}{d\Omega} = \frac{4\pi^2}{(2\lambda + 1)^2} \sum_\mu \frac{(\lambda + 1)^2 - \mu^2}{\lambda^2(2\lambda + 3)} \left| Y_{\lambda+1,\mu}\left(\frac{\pi}{2}, 0\right)\right|^2 \cdot |I_{\lambda+1,\mu}(\nu, \zeta)|^2 \cdot \cot^2 \nu/2 \cdot \sin^{-4} \nu/2;$$

and the total cross-section functions $f_{E\lambda}(\nu, \zeta)$ and $f_{M\lambda}(\nu, \zeta)$ given by

$$f_{E\lambda}(\zeta) = 2\pi \int_0^\pi \frac{df_{E\lambda}(\nu, \zeta)}{d\Omega} \sin \nu d\nu$$

$$f_{M\lambda}(\zeta) = 2\pi \int_0^\pi \frac{df_{M\lambda}(\nu, \zeta)}{d\Omega} \sin \nu d\nu.$$

The tables of the orbital integrals were obtained by numerical integration using the high-speed electronic computer BESK at Stockholm.

The orbital integrals were calculated for the following values of the parameters:

$$\nu = 10°(10°)180°$$

and

$$\zeta = 0.0(0.1)1.0(0.2)2.0, 4.0 \quad \text{for } \lambda = 1 \text{ and } 2$$
$$\zeta = 0.2(0.2)1.0(0.5)2.0, 4.0 \quad \text{for } \lambda = 3 \text{ and } 4,$$
$$\mu = -\lambda, -\lambda + 2, \cdots \lambda.$$

The tables are said to be accurate to the number of decimals given, which is six, except for small $\nu(10°)$.

The differential and total cross section functions are tabulated for the following parameter values:

$$\frac{df_{E\Lambda}(\nu, \zeta)}{d\Omega} \text{ and } f_{E\Lambda}(\zeta) \quad \text{for } \lambda = 1, 2, 3, \text{ and } 4$$

$$\frac{df_{M\Lambda}(\nu, \zeta)}{d\Omega} \text{ and } f_{M\Lambda}(\zeta) \quad \text{for } \lambda = 1 \text{ and } 2.$$

In the introduction to the tables properties of the orbital integrals are described. A review of the Coulomb excitation process wherein these integrals are discussed is given in the following publication: K. Alder, A. Bohr, T. Huus, B. Mottelson and A. Winther, *Rev. of Mod. Phys.* v. 28, 1956, p. 432.

Copies of these tables may be obtained from

Secretary
Det Kongelige Danske Videnskabernes Selskab
Dantes plads 5
Copenhagen V, Denmark,

at a cost of 10.00 kr.

LLOYD D. FOSDICK

University of Illinois
Urbana, Illinois

109[L, V].—JACK N. NIELSEN, *Tables of Characteristic Functions for Solving Boundary-Values Problems of the Wave Equation with Application to Supersonic Interference*, Report NACA TN 3873, 1957, 245 p., diagrams and Tables, 26 cm.

The present Technical Note deals with the problem of calculation of the pressure distribution on quasi-cylindrical bodies of revolution including the different possible cases of interference. The principles of the referring theory had been outlined in detail in several previous reports of Jack N. Nielsen and other authors (see References). This theory contains the so-called W_m-functions, which had been tabulated for different values of the quantities x (streamwise distance) and 1 (radial distance of unity corresponding to the body surface), W_0 occurring in the formula for pressure coefficient at the body surface, the higher order W-function being needed in connection with a calculation of interference pressure fields.

Nielsen stated some disadvantages of these $W_m(r, 1)$-functions, if the external pressure field (and not only the pressure at the body surface) was to be calculated. In this case singularities occurred and the results obtained as differences of two large numbers became inaccurate. Therefore a new method was developed based on a more general set of functions $W_m(x, r)$, r being the radial distance from body center line. Nielsen found that with this new function it is as easy to compute the pressures away from the body as those on the body.

The new $W_m(x, r)$ functions are defined as the inverse Laplace transformation of a function containing modified Bessel functions of the second kind:

$$W_m(x, r) = L^{-1}\left[e^{s(r-1)}\frac{K_m(sr)}{K_m'(s)} + \frac{1}{\sqrt{r}}\right].$$

They are the solution of the partial differential equation

$$\frac{\partial 2W_m}{\partial r^2} + \frac{1}{r}\frac{\partial W_m}{\partial r} - \frac{\partial 2W_m}{\partial \eta^2} - \frac{m^2}{r^2}W_m = 0$$

with the boundary condition

$$W_m = \frac{(1/8) - (m^2/r)}{r^{\frac{1}{2}}} + \frac{(3/8) + (m^2/r)}{r^{\frac{1}{2}}} \quad \text{at} \quad \eta = x - r + 1 = 0$$

$$\frac{\partial W_m}{\partial r} = 0 \quad \text{at} \quad r = 1.$$

The author discusses the different methods of evaluating this formulae particularly adapted to automatic computation. He found that the following 3 methods if used in a combined manner are to be considered as the most suitable one:

1. Power series in x with coefficients depending on m and r
2. Integral-equation methods
3. Characteristics using finite difference equations

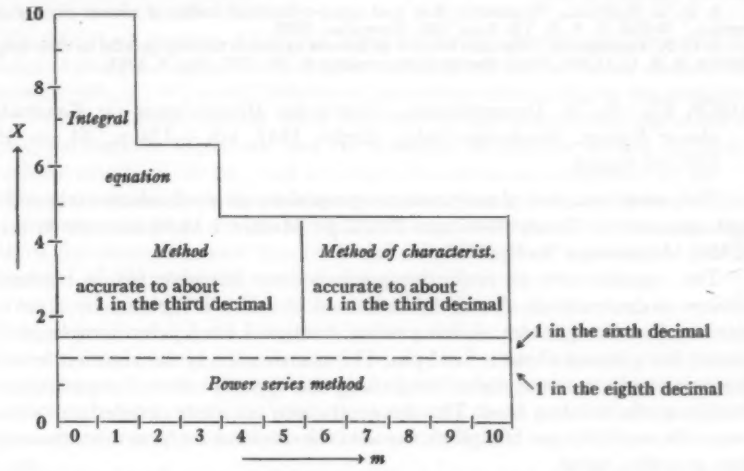
Thus the calculated tables of $W_m(x, r)$ cover a range of the order m from 1 to 10. The range of x is less for the higher order functions than the lower order functions because they become smaller faster. The actual ranges of the parameter x and r are

m	x	r (all values for each value m and r)
0 1	0 to 10	1, 1.1, 1.25, 1.5
2 3	0 to 2	2.0, 3.0, 4.0
4 5 6 7 8 9 10	0 to 5	6.0, 8.0, 10.0

On the whole 69,000 values are contained in the prescribed tables.

The author has intensively investigated the question of the accuracy and

came to the following results;



As to the physical significance of the $W_m(x, r)$ function, the author explains that these functions are to be considered as cylindrical pressure waves associated with a step in the radius of a streamwise quasi-cylindrical body.

The author finally interprets the application of the $W_m(x, r)$ functions to some aerodynamic examples which partly have hitherto not appeared in the literature, as

1) Calculation of the pressure field due to an indented body
2) Flow field of an infinite cylinder impulsively moved in axially direction in a viscous incompressible field
3) Wave drag of corrugated bodies
4) Arbitrary quasi-cylindrical body alone
5) Body-body or shock-body interference
6) Wing-body interference
7) Wortex-panel interference
8) Non-aerodynamic applications

The reviewed Technical Note represents without doubt a very remarkable contribution for a general solution of boundary value problems of the second kind involving the wave equation in three dimensions with approximately circular cylindrical boundaries or involving the unsteady heat-conduction equation in two space dimensions with nearly circular boundaries.

Reference is made on some publications which deal with the same or with similar problems; listed below are some of the most important.

ALFRED WALZ

National Bureau of Standards
Washington, D. C.

1. JACK N. NIELSEN, "Quasi-cylindrical theory of wing-body interference at supersonic speeds and comparison with experiment," NACA Report 1252, 1955.
2. JACK N. NIELSEN, "General theory of wave drag reduction for combinations employing quasi-cylindrical bodies with an application to swept-wing and body combinations," NACA TN 3722, 1956.

3. M. J. LIGHTHILL, "Supersonic flow past bodies of revolution," R. & M., No. 2003, British A. R. C., January 3, 1945.
4. D. G. RANDALL, "Supersonic flow past quasi-cylindrical bodies of almost circular cross-section," British R. A. E. TN Aero 2404, November, 1955.
5. G. K. BATCHELOR, "The skin friction on infinite cylinders moving parallel to their length," British A. R. C. 15,105, Fluid Motion Subcommittee F. M. 1772, Aug. 6, 1952.

110[P, Z].—W. W. DOBROWOLSKI, *Theorie der Mechanismen zur Konstruktion ebener Kurven*, Akademie-Verlag, Berlin, 1957, viii + 134 p., 24 cm. Price DM 19, bound.

This work was part of an extensive symposium on mechanisms and machines and appeared in Trudy Seminara Teorii po Mashin i Mekhanizmov 9, no. 36 (1950), Akademiya Nauk SSSR.

The constructions of projective geometry are implemented in mechanical devices to draw curves. A point constrained to a line is replaced by a pin constrained in a straight slot. A line passing through a fixed point is replaced by a slotted bar pivoted about a fixed pin. The transfer of angular rotation from one line to another is accomplished by locking the opposite sides of a parallelogram linkage to the rotating lines. The demonstrations are quite detailed and elementary with very little use of algebra. Some of the mechanisms have more theoretical than practical value.

A conic section can be drawn by a mechanism which simulates Pascal's theorem. If five points A, B, C, D, E of the conic are given, the variable sixth point M traces the conic. Fixed slots through BE and EC are used to guide pins which move, also, in three rotating slots which are hinged at H (the intersection of AB and CD) and points A and D. If the pin L lies on the triplet of lines (BE, HL, DL) and the pin K on (EC, HL, AK), then the pin M will be the intersection of AK and DL. Variations of the mechanism are shown corresponding to special cases of the theorem, namely, when any five elements are given (points and tangents).

Third and fourth degree curves are generated by compounding the simpler quadratic linkages, or by the use of four-bar and crosshead mechanisms. Mechanisms for quadratic transformations and Cremona transformations are described. Some higher degree curves, like the Lissajous figures, can be made by a combination of two functions of the same parameter. An inversor mechanism can convert a curve into another of higher degree.

MICHAEL GOLDBERG

Bureau of Ordnance
Navy Department
Washington, D. C.

111[S, X, Z].—S. FLUGGE, Editor, *Handbuch der Physik* (*Encyclopedia of Physics*), Band 1, Mathematische Methoden I (Mathematical Methods 1), Springer-Verlag, Berlin, 1956, vii + 364 p., 24.5 cm. Price DM 72.

[S, X, Z].—S. FLUGGE, Editor, *Handbuch der Physik* (*Encyclopedia of Physics*), Band 2, Mathematische Methoden II, Springer-Verlag, Berlin, 1955, vii + 520 p., 24.5 cm. Price DM 88.

The two volumes contain as complete a compendium of useful mathematical methods and theorems as one is ever likely to compile at any one time as part of an encyclopedia of physics. It would be impracticable to give more than a brief

indication of the contents and depth of the separate sections prepared by various authors.

The first three sections of Volume I were written by Josef Lense and cover (a) calculus, Lebesgue integration, complex variable theory, ordinary differential equations—89 pp.; (b) partial differential equations culminating in the classification of the semi-linear second order equation in two independent variables as elliptic, parabolic or hyperbolic and the Riemann method for solving the linear hyperbolic equation—30 pp.; and (c) elliptic functions and integrals—27 pp.

The next section on the special functions of mathematical physics was written by J. Meixner and gives a condensed but thorough development of the properties of all of the commonly used functions and relates them to the study of partial differential equations by the method of separation of variables—71 pp.

The final section of Volume I, by F. Schlögl, gives a detailed account of the techniques for studying the boundary value and eigenvalue problems for second order partial differential equations and includes the related aspects of the theory of integral equations and the calculus of variations (use is made of the Dirac delta function in studying the Green's function). Most of the material relates to elliptic equations and less space is devoted to the parabolic and hyperbolic equations—135 pp.

Volume II begins with a thorough treatment of selected topics in modern algebra by G. Falk which contains a discussion of linear algebra, the theory of group representations and the representation theory of algebras, with some brief indication of the application to mechanics—116 pp.

The second section is by H. Tietz and covers vectors, analytic geometry, projective geometry, differential geometry of curves and surfaces, Riemann geometry, tensor analysis, theory of spinors, contact and canonical transformations for a Hamiltonian system of equations—81 pp.

The next section by I. N. Sneddon is the only one written in English and is devoted to a detailed study of many integral transforms, their properties and their application to typical physical problems. It is supplemented by a discussion of Lebesgue integration, a development of the theory of linear operators and functionals in Hilbert and Banach spaces, and a short description of the theory of distributions (of L. Schwartz)—51 pp.

The section on numerical and graphical methods is written by L. Collatz. He discusses the operation of the slide rule, nomography, smoothing of data, approximations of functions, location of roots of polynomials and more general functions, solution of linear and non-linear systems of algebraic equations by direct and iterative methods, determination of eigenvalues of matrices; finite differences, interpolation and numerical integration; numerical methods for ordinary differential equations, initial value problems and mixed initial-boundary value problems for partial differential equations, methods of Ritz and Trefftz; integral and functional equations. The descriptions of methods are accompanied by adequately worked out examples—122 pp.

The last section of 28 pages is by H. Bückner and describes the operation and logical structure of analogue and digital computers and the coding of problems.

The above studies are of high quality and should serve as a valuable guide to the branches of mathematics that are most usable today in classical and modern

Physics. The treatments are generally self contained, but references to the mathematical literature are given to make it possible to keep the size of these volumes manageable. Mention is made of the fact that the subject of Probability Theory is not contained in these books. Otherwise the two books are a well-rounded and remarkably well coordinated set of chapters in what might be called Applied Mathematics in spite of its high purity.

E. I.

112[V].—J. J. STOKES, *Water Waves*, Interscience Publishers, New York, 1957, xxviii + 567 p., 23 cm. Price $12.00.

This book gives a lucid, well written, and connected account of the theory of wave motion in incompressible fluids with a free surface and subjected to gravitational and other forces. The author pays special attention to some of the more recent work in the field to which he has made significant contributions during and since World War II. He has taken considerable pains to supply background material in hydrodynamics, some of the mathematical tools needed and an extensive bibliography.

There are four main parts of the book: Part I, comprising Chapters 1 and 2, presents the basic hydrodynamic theory for non-viscous incompressible fluids. It also contains the derivation of the equations of the two principle approximate theories which form the basis for the discussion in the remainder of the book. The first of these is a consequence of the assumption that the wave amplitudes are small. The second comes from the assumption that the depth of the liquid is small. In both cases the approximate theories are derived from formal expansions with respect to a small dimensionless parameter and methods are given for determining the higher order terms. The derivation of the shallow water theory follows that first given by K. O. Friedrichs.

Part II, which consists of Chapters 3 to 9 inclusive, describes the linear theory obtained when it is assumed that the wave amplitudes are small. The main mathematical tool used is potential theory. This part is divided into three subsections: (A) dealing with wave motions that are simple harmonic oscillations in time; (B) dealing with unsteady or transient motions that arise from initial disturbances starting from rest; and (C) dealing with waves on a running stream.

At the end of subdivision A, the author outlines other methods for dealing with boundary problems which differ from those he discussed at some length earlier. In particular, the Wiener-Hopf method is briefly discussed, and the author "hazards the opinion that problems solvable by the Wiener-Hopf technique will, in general, prove to be solvable more easily by other methods. . . ." This reviewer is among those who do not share this opinion.

Part III, consisting of Chapters 10 and 11, deals with shallow water theory, the approximation resulting from the assumption that the depth of the liquid is small. This assumption leads to a system of non-linear partial differential analogous to the equations describing the motion of compressible gases.

The first part of Chapter 10 contains a detailed discussion of the theory of characteristics based on the discussion given by Courant and Friedrichs in their book, *Supersonic Flow and Shock Waves*. Simple waves and propagating discontinuities are discussed. Chapter 10 contains a detailed discussion of a variety of

concrete problems including conditions for the occurrence of a bore and a hydraulic jump, the motion resulting from the breaking of a dam, and the motion of frontal discontinuities in the atmosphere. It concludes with a discussion of some applications of the linearized version of the shallow water theory.

Chapter 11, entitled "Mathematical Hydraulics," deals with flows and wave motions in rivers and other open channels with rough sides. The fundamental equations derived in Chapter 10 are modified to include the force of resistance due to the rough sides which is taken to be proportional to the square of the velocity. This chapter contains an illustration of the length to which the author has gone to present portions of the book in such a way that they can be read, to a large extent, independently of the rest of the book. It contains a brief discussion of the method of characteristics which was given in Chapter 10; Figure 11.4.1, page 473, is identical with Figure 10.2.2, page 298.

A numerical method for calculating solutions of the differential equations is presented briefly. The method is based on the method of characteristics, but is applied in the x, t plane. The reader is referred to the paper of Courant, Isaacson and Rees for a discussion of the convergence of the method. No mention of stability questions is made.

The result of calculations for the prediction of an actual flood in the Ohio and at its junction with the Mississippi are reported on. A comparison of the predicted with the observed flood is also made. The author states "there is no doubt that this method of dealing with flood waves in rivers is entirely feasible since it gives accurate results without the necessity for unduly large amounts of expensive computing time on a machine such as the UNIVAC." The author also argues that such computations are cheaper and at least as good and as feasible as those made with the "analogue computers" previously used, namely scale models of long rivers or river systems.

Part IV, consisting of Chapter 12, discusses a few analytical solutions of the equations describing exact non-linear theory. This chapter deals with problems whose solutions can be expressed in power series. The book ends with an exposition of Levi-Civita's theory of progressive waves of finite amplitude in water of infinite depth which satisfy exactly the non-linear free surface conditions.

This stimulating book which contains a selection of material chosen because it interests the author provides an excellent treatment of many topics in the theory of incompressible flow. It will be useful to both students and research workers in the field.

A. H. T.

113[W, Z].—ANDREW D. BOOTH, L. BRANDWOOD, & J. P. CLEAVE, *Mechanical Resolution of Linguistic Problems*, Academic Press, Inc., New York, 1958, vii + 306 p., 21.5 cm. Price $9.80.

Modern electronic computers have been operating for more than ten years. Despite extensive experience with them and a host of innovations, little is known about their limitations.

It has been shown that some computers can play checkers, and some have played chess after a fashion, and some have even been forced reluctantly to learn a little. Each of these demonstrations throws some light on the ability of the

computer, and each of them up to now has resolved the question in favor of ability rather than of limitation. The exploration of these limits is an outstanding scientific problem.

Another demonstration which in addition promises to be productive of useful results is that of translation from one language to another. The first of several demonstrations was made in 1953 by International Business Machines and Professor Dostert of Georgetown University. Others have followed. But it still remains to be shown that the machines can turn out useful translations.

Professor Booth has been thinking of this problem since 1947, and he and his colleagues have put into this somewhat uneven book a resume of their work and ideas.

There is a chapter on APEXC, the machine they have at the University of London. This is a modest machine, inadequate for the large task of translating languages, and the fact that they have successfully used it to demonstrate many of the steps of translating is a tribute to the ingenuity of the authors.

The basic processes of making concordances and using dictionaries are described. Catalogues of sentence structures can be made in the same way.

There is a chapter on the analysis of the style of Plato, a problem of interest to linguists, historians, and philosophers. Plato's style changed as he grew older, with one striking discontinuity at about age sixty. This change in style enables scholars to divide the works into two groups, those earlier and those later than the change. Refinements in these studies of style lead to extensive burdensome drudgery at which a computer should excel. This is an excellently written chapter on a highly technical subject.

Getting back to translating words rather than counting them, there is a discussion of the mechanics of breaking words into stems and endings in order to have short dictionaries. Compound words, such as abound in technical German, can also be broken. If they are not broken, the size of the dictionary gets out of all bounds. Words with multiple translations, such as English "but" into German "aber" or "sondern," or "know" into "kennen" or "wissen," must be handled with regard to the context. The two words on either side are frequently enough for this purpose. The comparison of the expenses of word splitting and full dictionaries is confused.

The methods by which the computer could do the work of stem recognition, identification of parts of speech, and acting on the dictionary information about syntax are described in terms of analogous operations with punched cards, which are apparently thought to be more picturesque than the arcane operations of a computer. The most time-consuming operation will be that of consulting the dictionary. Considerable thought has been given to this, with the conclusion that the best way will be a modification of the bracketing method, in which the middle entry of the dictionary is consulted first, thereby establishing which half contains the word, then this operation is repeated on the proper half, and so on. In the introduction Booth says that this method was first given a practical form in 1955, but the reviewer is sure that versions were in use before that time. Some estimates of numbers of steps needed are made based on Zipf's law (attributed to Estoup). Some methods of compressing words to fit in small boxes are discussed, but nothing about the origins of these ideas is said, nor is any evaluation of their

usefulness made. The reviewer is under the impression that these are Russian ideas, and that they are probably not very useful. Ways in which the machine could resolve ambiguities by determining the subject matter are given. For instance a word which could be either "resistance" or "opposition" would be referred back to an earlier frequency count. This count of key words would determine that the subject was electricity, in which case the first would be used, or politics, whence the choice might be the latter. Very polished translations might be made by having a large collection of literary quotations, from which phrases of just the right flavor could be selected; to succeed this would require large storage and sophisticated programming, and a fast machine.

Braille II offers a fascinating opportunity to use the principles of mechanical translation against a simpler problem. Braille II uses a system of shorthand with a moderately complex set of rules, such as "do not use a contraction to bridge the components of compound words," or, some pairs of words can be treated as one word "where the sense permits." These rules can be reasonably well met with a dictionary of a thousand entries. Unfortunately—from the viewpoint of a computer enthusiast—this transcription is now being done by volunteers gratis, leaving no opportunity for the expensive machine.

Translation from French to English has been planned in some detail for APEXC. The small size of the memory permits only 250 French words in the vocabulary besides rules enabling the machine to recognize masculine from feminine, singular from plural, and most verb forms. Words are split into stems and endings. Not all the letters of a word need to be used; for example in *recevrai* only *rec* is needed to identify the stem and only *rai* the ending. If the vocabulary were more extensive more complication would be needed. There are plans for 46 verb endings. The authors assert that grammar and ambiguous endings are the only sources of real difficulty. No mention is made of how strange words will be handled. The procedures seem well thought-out, and sound as though they will work. We can look forward to seeing some examples of its output.

The chapter on translating German into English is quite in contrast to that on French. It is 162 pages long, more than half the book. It is not based on a machine experiment but only on an armchair study. Again in contrast with the chapter on French it looks toward the difficulties of the future rather than the opportunities of the present. The study is based on Oswald and Fletcher's "Proposals for the Mechanized Resolution of German Syntax Patterns" in *Modern Language Forum* XXXVI, 1951. Most of the examples of German used to illustrate the study are from a critique of Thomas Mann's novels and tend to emphasize the difficulties to be encountered rather than the infrequency of these unusual forms. One concludes that this untutored machine is to translate Mann at his deepest immediately or it will be considered a failure. Oswald and Fletcher have listed 82 syntactical units which can be used by a machine to analyze sentences. A second part of the study is the examination of a particular piece of text (7000 words) dealing with the specific subject of electron optics. Counts were made of the vocabulary and the syntactical rules which would need to be invoked. The number of words needed was 960. Each case of multimeaning, where in different contexts different English words are required, is taken up. Most of them are prepositions or adverbs. The variations in meaning are illustrated by *auf dem Tisch*,

auf dem Tanz, and *auf dem Lande*, in which *auf* is respectively "on," "at," and "in."

There is a resume of an article by I. S. Mukhin, *Proc I. E. E.* **103** (B), 463–472, 1956, telling of a machine program for translating English into Russian. Translation in this direction is more difficult than in the opposite, for the machine must synthesize the endings of the Russian words from the word order of the English. Each Russian word in the dictionary has 17 associated grammatical facts. After all these facts have been marshalled the Russian sentence is constructed. It may even happen that a word with its 17 grammatical properties is suppressed. If an English word is capable of more than one Russian equivalent, depending on context, this fact is recorded in the dictionary from which the machine (which is BESM) then has recourse to a second and more elaborate dictionary to resolve the ambiguity. Idioms are handled the same way.

A very short chapter discusses the proposal of a meta-language or intermediate language understood only by the machine, the object being to simplify translating among many languages. With such a meta-language translation from any one of N languages to any other could be done by $2N$ programs. Without it $N(N-1)$ might be necessary. The authors point out that if instead of a meta-language one of the N languages is selected to play a central role, then the number of programs is reduced to $2(N-1)$, smaller yet.

Last there is some discussion of a possible computer designed especially for translating languages. Armchair daydreaming about computers with peculiar properties is not without its rewards. It commonly brings out the fact that the modern computer is nearly optimal, and that abandoning any of its main properties for other advantages is rarely attractive. In this case the authors suppose that since the dictionary will be changed only rarely the ability to selectively alter the memory can be abandoned (perhaps only in part, although this point is not clear) in favor of a very large store. Using glass discs very large dictionaries (several million words) with small access times are economically feasible. It appears later in the discussion that the authors would like to be able to compile concordances and word counts; this would require a selectively alterable store. With a serial or serial-parallel machine words of variable length are feasible and at the same time certain economies become possible. The authors also recommend replacing multiplication and division by other instructions. One such replacement would be a transfer of control conditioned on a comparison of two words. Another would be to locate and copy in the accumulator an entry from the dictionary. An interesting suggestion would enable the machine to insert an additional word into a list of words which was already monotonic non-decreasing. This would have the new word in the register; it would then iterate the operation of copying the word at x into $x+1$ and then comparing it with the word in the register. If the word in the register is smaller continue with address x decreased by one; if it is large or equal copy from the register into $x+1$ and go to the next operation.

Despite its uneven quality this book is an interesting exposition. It has a lively approach and promises exciting advances for the future. With enough effort machines can not only translate but can translate adequately, or even better.

H. H. CAMPAIGNE

National Security Agency
Research and Development
Fort George G. Meade, Maryland

114[X].—R. A. BUCKINGHAM, *Numerical Methods*, G. P. Putnam's Sons, New York, 1957, xii + 597 p., 21.5 cm. Price $15.00.

This book, very comprehensive and very well written, is a combination of a reference book and a text book on standard numerical methods. The emphasis is on techniques and procedures, the proofs are as a rule omitted. The author gives a very detailed discussion of most of the methods available with abundant references to the literature and each important technique is followed by an example with all the necessary comments. Every chapter contains a collection of problems. Here is the list of topics: An introduction to computation; Lagrangian methods of interpolation, differentiation and integration; Newton's methods of divided differences; Differences at constant interval; Symbolic methods; Solution of ordinary differential equations; Polynomial and other algebraic equations; Fitting of data by the method of least squates; Direct solutions of simulaneous linear equations; Matrices and determinantal equations; Indirect methods for linear equations; Ordinary differential and Fredholm integral equations; Functions of two variables; and Partial differential equations. The book ends with useful appendices on: Relations between powers and factorials; Summary of difference formulas with remainder; Lagrangian formulas for differentations and integration; Orthogonal polynomials for curve fitting; Index on useful formulas in text.

The reviewer noticed the following omissions: a very incomplete discussion on ill conditioned matrices and an absence of warning that normal systems of equations have a tendency to be ill-conditioned and are almost certainly ill-conditioned for systems of order six or greater. Other omissions, like the Runge-Kutta method, are mentioned by the author in the preface.

This book, in the reviewer's opinion, is probably the best on the subject and is highly recommended not only for mathematicians but for all scientists with any interest in computing.

T. LESER

Ballistic Research Laboratories
Aberdeen Proving Ground, Maryland

115[X].—N. ARONSZAJN, A. DOUGLIS, & C. B. MORREY, JR., Editors, *Transactions of the Symposium on Partial Differential Equations*, Interscience Publishers, New York, 1956, vi + 334 p., 26 cm. Price $6.50.

The Program of the Berkeley Symposium on Partial Differential Equations, June 20–July 1, 1955, had two parts. One consisted of twenty hours of expository lectures on linear elliptic equations, linear hyperbolic equations, non-linear partial differential equations, and approximations to solutions of differential equations. The other was composed of thirty-two 45-minute lectures on a wide variety of topics. Twenty-eight of these papers, first published in *Communications on Pure and Applied Mathematics*, v. IX, No. 3, 1956, have been reissued in book form. A program of such length, with participants drawn from the leading centers of research in partial differential equations in the United States, not to mention contributions by a number of visiting scholars from abroad, would be expected to give a comprehensive picture of recent American acitivity in this field of mathematics. The book contains material to suit all interests, from casual beginners' to specialized experts', from proofs to conjectures. The subjects range from specific,

concrete, but delightful results obtained by simple, subtle devices from very special properties of the particular problems considered, to broad general conclusions derived by very abstract techniques.

To discuss all twenty-eight papers adequately in a short review would be impossible. Readers interested in numerical analysis and applied mathematics would find material of fairly immediate usefulness and value in at least half of the papers. The present discussion will arbitrarily be restricted to these, and even they will be described essentially by titles. Three papers are explicitly devoted to numerical analysis. John Todd has given elegant systematic stability analyses for several methods for approximating parabolic and hyperbolic differential equations. H. F. Weinberger and G. E. Forsythe have determined bounds for the eigenvalues of vibrating membrane problems in terms of eigenvalues for corresponding difference equations. L. E. Payne's new isoperimetric inequalities represent another approach to the basic problem of the two preceding papers. Many other papers deal with techniques and results useful for solving problems in applied mathematics. Noteworthy among these are A. E. Heins' excellent account of the scope and limitations of the method of Wiener and Hopf and J. J. Stoker's discussion of the determination of appropriate radiation conditions for steady state problems by passage for the limit in related nonsteady wave problems. Special representations of harmonic functions by S. Bergman, polyharmonic functions by A. Huber, and of singular and regular solutions of Cauchy problems for Euler-Poisson-Darboux equations by J. B. Diaz typify results that find frequent application. Information such as that about the nature of discontinuities of solutions of quasi-linear equations discussed by Y. W. Chen, about singularities of generalized axisymmetric potentials considered by Erdelyi, and several papers of estimates, growth, and local behavior of solutions has obvious importance for both pure and applied mathematics. In this connection, L. Bers' very lucid survey of local properties of solutions of elliptic partial differential equations deserves especial mention.

JOHN H. GIESE

Ballistic Research Laboratories
Aberdeen Proving Ground, Maryland

116[X].—F. B. HILDEBRAND, *Methods of Applied Mathematics*, Prentice-Hall, Inc., New York, 1956, xi + 523 p., 21 cm. Price $8.50.

"The principal aim of this volume is to place at the disposal of the engineer or physicist the basis of an intelligent working knowledge of a number of facts and techniques relevant to four fields of mathematics which usually are not treated in courses of the 'Advanced Calculus' type, but which are useful in varied fields of application. The text includes the result of a series of revisions of material originally prepared in mimeographed form for use at the Massachusetts Institute of Technology." The four fields are: Matrices, determinants and linear equations, Calculus of variations, Difference equations, and Integral equation; these are covered in chapters of about equal length. Each chapter is followed by an impressive collection of examples, about 100 to a chapter, and a list of references.

Readers who are mainly concerned with numerical analysis will find much useful material in this book. Some is not where it might be expected, e.g., the

extremal properties of characteristic numbers of a matrix is discussed in the examples.

The appearance of the book suffers from a very narrow margin.

JOHN TODD

California Institute of Technology
Pasadena, California

117[Z].—A. GIET, *ABACS or Nomograms*, Philosophical Library, New York, 1956, ix + 225 p., 22 cm. Price $12.00.

This is a translation of a book recently published in France and still available in a French edition [1]. Without sophistication it tells how to construct the simplest nomograms—Cartesian nomograms (essentially contours sketched on a plane with cartesian or other coordinates), alignment charts, and a few others. There is no real mathematical treatment, and the work can be followed by anyone with knowledge of graphs similar to that obtainable in our secondary schools.

The presentation is mainly by working out examples, each of which is stated as a formula pertaining to some shop or engineering problem.

The utilitarian nature of the book is illustrated by an appendix consisting of several sections. One concerns the advantages of Cartesian nomograms, and is largely concerned with their retention of accuracy even when the material on which they are drawn is stretched, deformed, or torn; this permits photography without worry about shrinking of paper during development of the print, for example. A second section of the appendix concerns advantages of alignment charts—accuracy, ease of reading, ease of interpolation, and so on. Another section lists several common forms of relations ($f_1 + f_2 + f_3 = f_1 f_2 f_3$ is one type listed, where the functions f_i are to be taken into account by suitable choice of scales) with recommended types of charts.

This is a useful text for those who want to construct nomograms, which may be the cheapest, the most convenient and the most widely used computational aids. Hence it has some considerable importance.

As noted above, it has no theory or bibliographic references. Readers requiring more theory would probably be happier with any of several books listed in the bibliography below, say [9], [10], [7] and possible [1]. Some bibliographical references are found in all these and in [5]. A few helpful tables are included in [3]. Otherwise, the present book is similar in content to those listed below, and if it is chosen by a reader this choice must be based on arrangment of material, wealth of examples (and many of the others listed below use the method of exposition by worked examples), or simplicity of exposition.

While the reviewer does not object to the utilitarian nature of the book, he would have been more favorably impressed if a few examples of the greatest attainments in the art of nomography has been cited; the common General Radio reactance charts relating inductance capacity, reactance and frequency for sinusoidal voltages [11] and similar charts have saved electrical engineers enough time to pay for all the world's research on nomographs, and the nomograms by Rybner [12] are examples of a professional nomographer of a high level of competence attacking a problem of seeming forbidding extent. It seems unfortunate that such examples are not made available to the student reading Giet's book.

An abstract of the table of contents follows; there is no index.

I. Relations Between Two Variables

1. Diagrams
2. Scales

II. Cartesian ABACS

3. Relations Involving Three Variables
4. Superimposed Cartesian ABACS for Three Variables
5. Superimposed Cartesian ABACS for Four Variables
6. Superimposed Cartesian ABACS for Relations Involving n Variables ($n > 4$)

III. Alignment Charts

7. Parallel Coordinates
8. Graphical Representation of a Relation of the First Degree Between Three Variables
9. Standard Forms of Alignment Chart for Relations Between Three Variables
10. Charts for Four Variables
11. Charts for Three Variables Requiring the Use of Auxiliary Variables
12. Combined Charts for Several Equations Involving Common Variables

IV. Alignment Charts Not Based on Parallel Coordinates

13. Charts with Concurrent Scales
14. Circular ABACS
15. Double-Alignment Charts for a Relation Between Four Variables of the Form $f_1f_2 = f_3f_4$

V. Relations Between n Variables ($n > 4$)

16. Relations of the Form $f_1 + f_2 + f_3 + \cdots + f_n = 0$
17. Relations of the Form $f_1 \cdot f_2 \cdot f_3 \cdots f_{n-1} = f_n$

Appendix

Choice of Methods: Choice of Type of ABAC

C. B. T.

1. W. J. Allcock & J. R. Jones, *The Nomogram*, Pitman, London, 1946.
2. S. Brodetsky, *A First Course in Nomography*, Bell, London, 1920.
3. D. S. Dairs, *Empirical Equation and Nomography*, McGraw-Hill, New York, 1942.
4. Maurice Kraitchik, *Alignment Charts*, van Nostrand, New York, 1944.
5. A. S. Levens, *Nomography*, John Wiley and Son, New York, 1948.
6. M. Lien, *Nomografi*, Karlebo, Stockholm, 1945.
7. J. Lipka, *Graphical and Mechanical Computation*, John Wiley and Son, New York, 1918.
8. C. O. Mackey, *Graphical Solutions*, John Wiley and Son, New York, 2nd edition, 1944.
9. W. Meyer zur Capellen, *Leitfaden der Nomographie*, Springer, Berlin, 1953.
10. M. W. Pentkowski, *Nomographic*, Akademie, Berlin, 1953.
11. F. E. Terman, *Radio Engineers' Handbook*, McGraw-Hill, New York, 1943.
12. J. Rybner, *Nomograms of Complex Hyperbolic Functions*, Gjellerups, Copenhagen, 2nd edition, 1955. [*MTAC*, v. 11, 1957, Rev. 80, p. 207.]

118[Z].—T. E. IVALL, Editor, *Electronic Computers*, Philosophical Library, Inc., New York, viii + 167 p., 22 cm. Price $10.00.

This book is a non-mathematical introduction to the principles and applications of computers employing tubes and other electronic devices. The treatment has been made very general in order to provide a broad background of the entire field of computing.

The first chapter of the book discusses some of the outstanding contributions made to the computing field over the past three hundred years.

The second chapter discusses the general principles of computing. Comparisons are made between pencil and paper, desk calculator, and electronic computer methods, in doing a particular type of mathematical operation.

The next three chapters are devoted to analogue computers. Typical circuits of analogue computers are explained in general terms. Descriptions are given on how the circuits operate, how they are constructed, and combined to fulfill the requirements for a particular application. The types of inputs required, scaling, types of outputs and some of the applications for which an analogue computer are best suited are described.

The remaining six chapters are devoted to digital computers. The basic arithmetic operations are discussed along with some of the different types of circuits required to perform these operations. Storage systems such as magnetic drums and discs, magnetic tape, magnetic core arrays, and cathode ray tubes are discussed and comparisons are made between the relative speed of the different systems.

The applications of digital computers to data processing, machine tool control, production planning are discussed in general terms.

The book gives the reader a painless introduction to the principles of operation and application of high-speed electronic computers and computer components. A background in mathematics or a previous knowledge of computer circuitry is not required for a thorough understanding of the contents. In clear and concise terms, the author covers the subject in eleven well-written chapters. The book is written primarily for electronic technicians and students. Engineers in other fields who wish to become acquainted with the subject of computers will find the text both interesting and informative. The book lays a firm foundation for subsequent detailed reading in the digital computer field. Persons in the sales, managerial, administrative, and executive fields would find it quite helpful. It is not intended, however, that the book be of use to engineers who are well informed in computing techniques and equipment. The excellent illustrations of components and circuitry serve well as an aid to comprehension.

CHESTER H. WALLIN

Computing Laboratory, BRL
Aberdeen Proving Ground, Maryland

119[Z].—R. K. LIVESLEY, *Digital Computers*, Cambridge University Press, London, 1957, viii + 53 p., 21 cm. Price $1.75.

This slim volume of 53 pages is published in the Cambridge engineering series. It amounts to a brief introduction to digital computers for engineers who are prospective users of machine. The author starts by analyzing a simple calculation

(the solution of three simultaneous linear algebraic equations in three unknowns) to show what arithmetical and logical steps are involved. He then describes a simple computer which could carry out this calculation. Next a brief and sketchy treatment is given of the physical equipment needed for memory and input-output units; no attempt is made to show how control and information processing functions are actually carried out. There follows a chapter on programming, stressing the utility of subroutines and showing how they can be organized into complete programs. The book is concluded by a chapter giving a brief sketch of typical engineering problems requiring extensive computations.

The writing is clear, unpretentious, and easy to follow. The book could be understood, with a little additional explanation, by an interested reader of limited scientific background.

C. V. L. S.

120[Z].—G. A. MONTGOMERIE, *Digital Calculating Machines*, Blackie and Son Limited, Glasgow, Scotland, 1956, vii + 262 p., 22 cm. Price 30s. net.

This is an elementary practical guide without pretentions of scholarship and without efforts to romanticize computers. The material covered includes all common types of computers, but much is sparse, particularly with regard to larger computers. Illustrations of common machines are included, but nothing like complete coverage of machines available (particularly those easily available outside but not within the United Kingdom) is attempted.

Additional references are given at the end of each chapter; the author's view that reference material for most topics is not useful or is hard to obtain is not entirely shared by the reviewer, but, on the other hand, it seems to the reviewer that by the time a reader has advanced to a point where he needs such references they may become known through other contacts. In any case the bibliographic references are not complete.

In short, what is put together is an elementary work giving the first principles of operation of the various types of machines discussed. It is a practical work, although it does not group and exhibit recipes for particular calculations. Square roots are taken by repeated subtraction on page 95, for example, with a suggestion that one of the methods described later in the book be used for most work. On page 117, square roots are again mentioned and logarithms are suggested. Finally, on page 141 the common iteration, $x_n = \frac{1}{2}(x_{n-1} + a/x_{n-1})$ is introduced as the best way to find $a^{\frac{1}{2}}$, but never is there given a rule for continuing by this method an extraction started by successive subtraction, which is easily brought about on some machines. Still, such methods are usually (and properly) brought to users' attention by the manufacturers of the various machines, and description in a small book such as the present with no reference to a laboratory instrument would entail difficult exposition.

There is an index of processes treated, on pages 143–144, and this has some of the uses of a set of recipes.

Analysis of strong and weak points of machines and their applications like that in [1] is missing almost entirely. The Mechanical principles are treated, but not deeply, or to the extent some are described in [2].

Thus, what is put together is a sound but not advanced text, neither deep nor

encyclopedic in nature, but containing material not easily available elsewhere in English. It is mainly useful in connection with manually operated machines, here mainly presenting a description for those who have not yet used such machines to any great extent. It describes many but not all types of machines. There is undoubtedly a set of readers who will find such a text more useful than [2], [3], [4], [5] or [6], for example.

The table of contents follows:

Chapters

I. Introduction
II. Key-responsive Adding Machines (Group I, Type 1)
III. Key-set Adding Machines (Group I, Type 2)
IV. Four-rule Machines, Lever-set, Hand-operated (Group I, Type 3)
V. Four-rule Machines, Key-set and Electrically-operated (Group I, Type 3, cont.)
VI. Four-rule Automatic Machines (Group I, Type 4)
VII. Computing with Four-rule Machines
VIII. Punched-card Machines (Group II)
IX. Special-purpose Machines (Group III)
X. Universal Sequence-controlled Machines (Group IV)
XI. Electronic Sequence-controlled Machines (Group IV cont.)
XII. Programming of Sequence-controlled Machines

C. B. T.

1. L. J. Comrie, "The application of commercial calculating machines to scientific computing," *MTAC*, v. 2, 1946, p. 149–159.
2. F. J. Murray, *The Theory of Mathematical Machines*, King's Crown, New York, 1947 and (revised) 1948.
3. F. A. Willers, *Mathematische Maschines and Instrumente*, Akademie, Berlin, 1951.
4. F. A. Willers, *Mathematisch Instrumente*, Gruyter, Berlin, 1926.
5. W. Meyer zur Capellen, *Mathematische Instrumente*, Akademie, Leipzig, 1949.
6. E. Hersburgh, *Modern Instruments and Methods of Calculation*, Bell, London, undated.

121[Z].—Edmund C. Berkeley & Lawrence Wainwright, *Computers, Their Operation and Applications*, Reinhold Publishing Corporation, New York, 1956, ix + 366 p., 22 cm. Price $8.00.

The authors admit in their preface that it is a hopeless task to gather all the information about computing machinery. Their approach is to "do the best one can to report information already learned—incomplete as it is—indicating areas where other information is needed." The appeal is to the general reader, and it is truly unfortunate that the book is so disorganized as to be confusing to the novice in the field.

One assumes from the first chapters that the approach is for the non-expert, yet the section on analog equipment (written by the second author) does not hesitate to use derivatives and other mathematical operators with the assumption of previous knowledge on the reader's part.

The peculiar mixture of the general and the specific which is the hallmark of the first author leads him to make some misleading and some inaccurate statements. A few quotations will perhaps suffice to show what is meant.

On the operation of overflow he states on page 41: "Another useful operation

is *overflow*, in a counter; this is the production of a number in the counter which is beyond its capacity, giving rise to a carry signal that can be employed for useful purposes. This would happen for example in a six-place counter holding 999784 if 4000 were added."

Unfortunately, he says nothing more about the usefulness of overflow, nor what purposes the carry signal could be employed for. In addition, the quotation serves to mar the distinction between adders and counters, which he does not clarify elsewhere.

On accuracy he states on page 58: "An automatic digital computer is surprisingly accurate. One of the earliest machines, the Harvard IBM Mark I, represented numbers of 23 decimal digits. . . ." This, by the author's own definitions in the glossary appended to the book, is not accuracy, but precision. A few sentences further on he repeats the error: "Accuracy to such a high degree is not often useful." Again it is precision which is meant. Finally to conclude this exhibition, and make "confusion worse confounded," we have: "One machine, the IBM Electronic Selective Sequence Calculator, . . . calculated the motion of the moon with new and remarkable *accuracy*, locating the moon *precisely* in the heavens at six-hour intervals for four centuries." (Italics are the reviewer's.)

In Section IV he deals with types of automatic computing machines that are not digital or analog computers. In drawing this artificial distinction he only succeeds in completely confusing the reader. He lists, among other non-computers on page 162, an astronomical telescope aiming equipment to adjust the direction of a telescope to follow a particular star field. This is as clear an example of an analog computer as one would wish, and the author himself says so on page 293: "And automatic analog computers are responsible for keeping a telescope pointed at a heavenly body while the astronomer watches it through the eyepiece."

It would have been far better to avoid such artificial distinctions if he was not able to be consistent in his definitions. Such loose thinking could be demonstrated by many more references to the text.

Section III, written by Lawrence Wainwright, is, by contrast, "a good deed in a naughty world." It is a clear elementary exposition of the functions and elements of analog computers, with examples of various types. The book would be worthwhile if it were all on a par with this section.

Section VII, on applications, spends a good deal of space on arguments as to the usefulness of computers in fields where they have long made their mark. The actual descriptions of applications in some fields are very terse. In mathematics, the whole field of numerical analysis is ignored. Computers, according to this author, have determined whether numbers are prime, and calculated the values of π and e to 2000 decimal places. The basic importance of being able to ask the computer the right questions so as to get the right answers is nowhere stressed. Even the fact that computers have opened a whole new realm of unsolved problems is glossed over.

The book concludes with a section of references, a roster of organizations and computing services, and a glossary. The index reveals no mention of the name of von Neumann, an omission which the author should find it hard to justify.

I have rarely read a book which could make me angry by the mere style of the author; this one does. I have tried to approach it with the objectivity of a

person new to the subject, as a general reader might. Even this did not help; the fuzzy thinking obscured all the difficult parts, and the "cute" analogies which compare a computer to a large railroad yard with switching stations are of no avail in the serious business of understanding what a computer really is.

F. HOLLANDER

University of California
Los Angeles, California

122[Z].—GEORGE F. FORBES, *Digital Differential Analyzers, An Applications Manual for Digital and Bush Type Differential Analyzers*, Third Edition, Published by the author, Pacoima, California, 1956, xii + 154 + 19 p. of Appendices and 9 p. of index sheets, bound, soft cover. Price $7.50.

[Z].—GEORGE F. FORBES, *Digital Differential Analyzers, An Applications Manual for Digital and Bush Type Differential Analyzers*, Fourth Edition, Published by the author, Pacoima, California, 1957, iii + iv + 154 + 22 p. of Appendices and 8 p. Index sheets. Price $5.00.

This book is strictly as described in the title, an applications manual. A very brief introduction to the principles of digital differential analyzers is given in Chapters I and II, not sufficient to explain their operation fully to one unacquainted with devices of this type. However, this is not a serious defect, since the detailed information on the use of these machines contained in the remaining twenty-seven chapters will only be of interest to those who really understand them anyway. One's general impression is that the author presents this material carefully and precisely, and that it must be very useful to the audience to whom it is addressed.

The Third and Fourth Editions are essentially identical. The author states that some few corrections have been made in the Fourth, while the only additions are those of a few extra Exercises and a few extra items in the Bibliography.

C. V. L. S.

123[Z].—O. J. DAHL, *Autocoding for the Ferranti Mercury Computer (MAC)*, NDRE Report No. 24, Oslo, Norway, 1957, iii + 92 + 11 p. of appendices.

This is the description of the external features of an algebraic compiling system similar to FORTRAN written for the Ferranti Mercury Computer. In addition to the floating-point representation of numbers which is built into the machine, the system allows arithmetic operations on complex numbers, double-length floating-point numbers, and short integers. Programs are written as sequential lists of directives, or instructions, which may themselves be elaborate algebraic formulas composed of pre-defined variables and constants. For example, the directive to compute

$$x/2(1 + x^2)^{\frac{1}{2}} + \tfrac{1}{2} \ln (x + (1 + x^2)^{\frac{1}{2}})$$

is programmed as

$$(X/2)Q(1 + X*2) + (/2)\text{IEXP}(X + Q(1 + X*2)) \rightarrow Y$$

where X denotes a single variable previously defined, Q the square root, IEXP the inverse exponential, * the power, + addition, and / division. Functions available directly in the symbolism are exponential, sin, cos, tan, square root and their inverses as well as more complicated functions such as integral and derivative. Other functions can be incorporated as subroutines.

The built-in B-registers are not referred to directly; a postfix method of indexing sets of variables is used. For example, the matrix element a_{ij} is written as An12 where index nl is used to count rows and n2 to count columns. For all programs the extents of fields of variables must be defined in a set of field foundations preceding the instructions.

The Mercury high-speed store consists of 16 pages of 32 floating-point numbers each. This is backed up by a magnetic drum from which a page at a time may be transferred. Large matrices, for example, must be stored on the drum and transferred as needed to the high-speed store. Matrices are frequently filled out to multiples of 32 by adding zero elements. A field in the H. S. store is referred to by a symbol such as FA, on the drum as FMA. Operations on an entire field in the H. S. store are possible without counting. For example, the directive 0→FA sets the whole field FA to zero. Facilities are also available for constructing compound logical statements. In the report, programs are shown for various matrix and vector operations, the solution of linear equations by the Jordan-elimination method, and there is a discussion of indexing for non-rectangular fields.

The normal method of looping, using counters and conditional jump instructions, can be used but a more sophisticated method, using operators, is also described. These operators cause the sequence of directives within their range of influence to be executed systematically for all possible values of an index stated in the operator.

The system is said to prepare programs comparable to manually coded ones. Storage allocation of program and data is still the responsibility of the programmer if these cannot all be contained in the H. S. store. No indication of experience with the system is given nor is there very much information about the mechanism of compilation. It appears to be a comprehensive scheme admirably suited to the needs of a seasoned programmer. The more elementary sections of the report would undoubtedly permit even the beginner to use the system.

J. N. P. HUME

University of Toronto
Toronto, Canada

124[Z].—O. J. DAHL, *Multiple Index Countings on the Ferranti Mercury Computer*, NDRE Report No. 23, Norwegian Defence Research Est., Oslo, Norway, 1957, iv + 43 + 10 p. of diagrams.

This report describes a systematic method of handling multiple countings in programs for problems such as linear algebra or tensor calculus. The form of counting depends on how a set, or field, of data is stored in the machine. The dimensionality of a field stored completely in the high-speed store is the number of indices describing an element of the field, e.g., 1 for a vector, 2 for a matrix. The range of an index is known as an extent of the field. In general an extent may

be a function of other indices; fields whose extents are constant are rectangular fields. The H. S. store of a computer is essentially a one-dimensioned field and the problem as stated is "to store an n-dimensioned field in a linear one." For example, matrices can be stored as sets of linear subfields, end-to-end, e.g., row by row. Use of the secondary store, the magnetic drum, can increase the number of indices. For example, if a single column of the matrix occupies several sectors of the drum, countings must proceed both on sectors and elements within sectors. Various methods of partitioning matrices are discussed. Flow diagrams and Mercury Computer Programs are given for a variety of partitioning methods. In fitting a field into the machine storage, when it cannot be divided into an integral number of similar subfields, zeros may be added to complete subfields. It is found best to complete at least all H.S. store subfields with zeros allowing for incompleteness on the drum only. Countings may avoid operations on the zeros but the gain in time for large matrices appears to be offset by the higher degree of complexity in the innermost countings.

J. N. P. HUME

University of Toronto
Toronto, Canada

125[Z].—LORENZO LUNELLI, "Un comando di computo di cicli per una calcolatrice elettronica aritmetica a tre indirizzi," *La Ricerca Scientifica*, v. 27, 1957, p. 3381–3394.

This paper describes a "cycle computing instruction" added to the CRC-102 A digital computer located at the Centro di Calcoli Numerici of the Milan Polytechnical School. Essentially, the command combines counter modification (by subtraction of binary one from specified sections of the contents of a counter cell—any desired cell of the memory) with branch point on zero results in any section so specified. The feature of interest in the command is the implicit capability of combining several separate counts in a single cell with branching possible on each such count. The command is mechanized as follows: Let the command be represented by $Cm_1m_2m_3$. Then, m_1 is the counter cell; (m_2) specifies the division of (m_1) into separate counters by locating contiguous binary ones in the corresponding bit positions of (m_2) (individual counters are therefore separated by at least one bit); the command subtracts one from the least significant bit of each of those sections of (m_1) for which (m_2) has such a contiguous sequence of binary ones and replaces (m_1) by the result; m_3 specifies the location of the next command if none of the sections so specified becomes all zero; otherwise the next command is taken from the address following that of the command being executed. The use of this command is particularly effective for address modification in a repetitive subroutine. The paper includes illustrations of this type of operation.

An appendix to the paper describes a proposed command with somewhat similar purpose but different mechanization. The proposed command is similar in function to the familiar "B-box" except that it includes the modification of the B-box with the actual augmentation of the command, rather than with the branching decision. The proposed mechanization is as follows: Let the command be $Cm_1m_2m_3$; then m_1 is the location of the command to be executed; m_2 is the

"B-box," and its contents modify (m_1); (m_2) is the modifier of (m_2); the command results in the execution of the command stored in m_1 as modified by (m_2) and in the replacement of the original contents of m_2 by the result of $(m_2) + (m_3)$.

No time estimates are given for either of the two commands. However, the first one appears to be fairly cheap, requiring perhaps one or two word times more than a normal subtract command. On the other hand, the second, proposed, command requires four memory look-ups in addition to those of the command being modified and executed as part of the operation. Under many circumstances this would be very costly in time. These commands are quite representative of some of the complex built-in sub-routines which can fairly easily be added to a micro-programmed machine such as the CRC-102A. Such built-in sub-routines are much faster than the corresponding programmed sub-routine. On the other hand, it is characteristic of the micro-programmed approach that the resulting built-in sub-routines are slower than they would be if designed otherwise; the advantage of micro-programming is the relatively efficient use of equipment.

R. M. HAYES

Business Systems Department
Magnavox Research Laboratory
Los Angeles, California

TABLE ERRATA

265. EDWIN P. ADAMS, *Smithsonian Mathematical Formulae and Tables of Elliptic Functions*, Second reprint, The Smithsonian Institute, Washington, D. C., 1947. See also *MTAC*, v. 1, p. 191, 325; v. 2, p. 46, 352–3; v. 3, p. 314, 423; v. 6, p. 236.

P. 122, formula **6.42,** no. 4, the sign before the $\sum$ should be $-$.

P. 127, formula **6.475,** no. 2, the formula is correctly printed in this reprint. See *MTAC*, v. 2, p. 46.

P. 139, formula **6.821,** in the numerator of the fraction in the second summation (involving sines), *for* 1, *read* n.

P. 139, formula **6.822,** the formula should read

$$e^{cx} = \frac{2c}{\pi}\left\{\frac{e^{c\pi}-1}{2c^2} - \sum_{n=1}^{\infty}\frac{1-(-1)^n e^{c\pi}}{c^2+n^2}\cos nx\right\} \quad [0 \leqq x \leqq \pi]$$

E. G. H. COMFORT

Ripon College
Ripon, Wisconsin

266. F. J. MASSEY, JR., "Distribution table for the deviation between two sample cumulatives," *Annals of Math. Stat.*, v. 23, 1952, p. 435–441.

The entire table was recomputed using the IBM 701 electronic computer at the University of California's Computer Center, Berkeley, California. In addition to the following 46 errors there are approximately twenty other entries in which the last digit has been rounded incorrectly. The 16 corrections marked with an

asterisk were previously reported by J. L. Hodges, Jr., in *Arkiv för Matematik*, 1958, in print.

N_1	N_2	h	For α	Read α
3	7	5	.00833	.01677
6	7	35	.99126	.99184*
6	8	20	.99467	.99534*
7	8	16	.18943	.20140
		25	.65703	.68687*
		26	.71795	.74779*
		27	.77762	.80746*
		34	.93629	.94406*
3	10	20	.83566	.86014
4	10	4	.01518	.01598
7	10	29	.68511	.67935
		49	.98416	.98601*
		50	.98900	.99054*
		53	.99393	.99548*
8	10	16	.67045	.70447
		21	.90123	.90489*
		22	.92317	.92957*
9	10	26	.35336	.36252
		53	.96564	.96973*
		62	.99281	.99296*
9	12	14	.70465	.73189
9	15	30	.99448	.99519
		31	.99637	.99707
		32	.99731	.99801
		33	.99799	.99870
		34	.99864	.99934
10	15	12	.82837	.82715
12	15	21	.72675	.73230
		22	.76664	.77734
8	16	5	.57501	.57581
		6	.76578	.75919
12	16	14	.53765	.55837
		29	.99423	.99475*
		30	.99624	.99669*
12	18	12	.73908	.73811
10	20	2	.00020	.00197
12	20	30	.97431	.97511
		31	.98093	.98173
		32	.98554	.98635
		33	.98915	.98996
		34	.99216	.99298
15	20	27	.96528	.96531
		29	.98044	.98101
		30	.98586	.98637*
16	20	33	.94026	.94023
10	40	7	.12214	.12114

LOUIS S. DAVIS

Statistical Laboratory
University of California
Berkeley, California

NOTES

Association Francaise de Calcul

The editors have been informed of the founding of the Association Francaise de Calcul. The first President is the well known astronomer Andre Danjon, of the French Academy of Sciences.

The Association intends to develop all aspects of numerical analysis, computing and associated fields; its members will include people interested in these topics from any profession—engineers, economists, statisticians and so on.

The address of the new Association is:

Association Francaise de Calcul
Institut d'Astrophysique
98 bis, Boulevard Argo
Paris 14, France

C. B. T.

Milton Abramowitz (1915–1958)

Dr. Milton Abramowitz, head of the Computation Laboratory of the Applied Mathematics Division at the National Bureau of Standards, died suddenly of a heart attack at his home on Saturday, July 5.

Dr. Abramowitz was a valued younger scientist of the Bureau. Starting in 1938 as an original member of the Technical Planning Staff of the New York Mathematical Tables Project, the scientific program of which was under the direction of the Bureau, he moved upward to become in 1954 the Chief of the Bureau's Computation Laboratory. In addition to the supervision of the Laboratory's computation facilities, he directed the preparation of mathematical tables and performed research in numerical analysis and mathematical physics. Among his many accomplishments, he is known for his significant contribution to the planning and preparation of the well-known volumes of basic mathematical tables published in the Mathematical Tables Series of the original New York group and in the Applied Mathematics Series of the Bureau, and most recently for his editorship of a comprehensive *Handbook of Mathematical Functions*, which the Bureau is preparing under the sponsorship of the National Science Foundation.

Born in Brooklyn, New York in 1915, Dr. Abramowitz received his B.A. in mathematics in 1937, and his M.A. in 1940, both from Brooklyn College. He earned his Ph.D. from New York University in 1948.

Dr. Abramowitz received the Gold Medal in mathematics from Brooklyn College in 1933, and was awarded the Meritorious Civilian Service Award from the U. S. Navy in 1945. He was a member of the American Mathematics Society, and the Washington Academy of Sciences.

Horace Scudder Uhler

1872–1956

By J. W. Wrench

The art of highly accurate computation lost one of its most gifted exponents and practitioners when Professor Horace S. Uhler died in Meriden, Connecticut on December 6, 1956, following a brief illness.

Professor Uhler was born in Baltimore, Maryland on August 5, 1872, and, following his education as a physicist, he spent all his adult life in the academic world. He received a baccalaureate degree from Johns Hopkins University in 1894, following which he was an instructor in the Academy of Northwestern University, laboratory director at the University School in Baltimore, and graduate student and fellow at Johns Hopkins, where he received his doctorate in 1905.

The following year he served as Carnegie assistant in physical chemistry at Johns Hopkins University, whence he was appointed instructor in physics at Yale College, Yale University. This was the beginning of an association that was to extend throughout the rest of his life, except for an interlude, 1925–26, when he was the director of the department of physics at Gettysburg College. In 1937 he received an honorary master's degree from Yale University, and concurrently he was elevated to the rank of professor of physics.

The title of Professor Emeritus of Physics was officially conferred on Professor Uhler on the occasion of his retirement in 1941. Enlistment of many of the younger members of the faculty during the second World War led to his service as a special lecturer in physics for a period of two years. After returning to retirement in 1943, he devoted all his energies to his avocation, mathematical computation, which to a progressive extent had occupied his attention during the preceding half-dozen years.

He was pre-eminent in the field of elaborate computation because of his scrupulous attention to each detail and because of his extraordinary efforts to insure the impeccable quality of his published results. Professor D. H. Lehmer once referred to Uhler's "indefatigable perspicacity" in describing his practice of subjecting all his publications to the most careful scrutiny. This habit led to the occasional discomfort of editors who did not always meet his typographical standards of perfection!

Uhler's numerical research focused principally on factorials and their reciprocals, natural logarithms, and Mersenne numbers, the computations being performed by means of electrically operated desk calculators.

His exact values of factorials, which appeared in a succession of publications, and his elaborate table of reciprocals of factorials to 475D remain unrivalled to the present time. A definitive compilation of radix logarithms is his 137D table of such data, privately published in 1942. Supplementary material is available in a series of papers giving approximations (ranging from 155D to 330D) to the natural logarithms of small primes, as well as to the modulus of the common system of logarithms and to π.

At the suggestion of Professor R. C. Archibald he began in 1944 the pioneer investigation of the primality of the six remaining Mersenne numbers $2^p - 1$, corresponding to $p = 157, 167, 193, 199, 227$, and 229, for which no identification was then available. His numerical proofs that these integers are all composite were confirmed by subsequent calculations performed on the National Bureau of Standards Western Automatic Computer (SWAC).

A sequel to this investigation was the computation of the first seventeen perfect numbers, following the identification by SWAC of all Mersenne primes for exponents p not exceeding 2303.

A chronological list of Professor Uhler's significant mathematical publications is appended. Copies of certain unpublished manuscripts have been deposited in

the library of Brown University. These include a table of factorials of all integers from 201 to 300, inclusive, and a table of factorials decomposed into prime factors for consecutive integer arguments ranging from 1001 to 3400, inclusive. The latter constitutes an elaborate extension of a table of M. Alliaume published in 1928.

In addition to his numerous papers in mathematics, Professor Uhler contributed a number of papers in the fields of geometrical optics, spectroscopy, and x-rays.

He was a charter member of the Mathematical Association of America, a member of the American Mathematical Society, the London Mathematical Society, the Optical Society of America, a life member of the Société Française de Physique, and a fellow of the American Association for the Advancement of Science, and the American Physical Society. He was a former associate editor of the *American Mathematical Monthly*, the *Physical Review*, the *American Journal of Science*, and the *Journal of the Optical Society of America*.

A portrait of Professor Uhler, accompanied by biographical and bibliographical information, appears in the book *Mathematical Table Makers*, by R. C. Archibald, published by *Scripta Mathematica*, New York, New York, 1948.

The indelible impression left on the memory of the writer, who enjoyed Professor Uhler's friendship for the past two decades, was that of a genuine scholar of broad interests and attainments, who set for himself the highest standards of clarity and accuracy in his research.

1. "On the numerical value of i^i," *American Mathematical Monthly*, v. 28, 1921, p. 114–116.
2. "A new table of reciprocals of factorials and some derived numbers," Conn. Acad. of Arts and Science, *Trans.*, v. 32, 1937, p. 381–434.
3. "Log π and other basic constants," Nat. Acad. Sci., *Proc.*, v. 24, 1938, p. 23–30. [See also *MTAC*, v. 1, 1943, RMT **95**, p. 55.]
4. "Recalculation and extension of the modulus and of the logarithms of 2, 3, 5, 7, and 17," National Acad. Sci., *Proc.*, v. 26, 1940, p. 205–212. [See *MTAC*, v. 1, 1943, RMT **96**, p. 56.]
5. "The coefficients of Stirling's series for $\log \Gamma(x)$," Nat. Acad. Sci., *Proc.*, v. 28, 1942, p. 59–62. [See *MTAC*, v. 1, 1943, RMT **97**, p. 56.]
6. *Original Tables to 137 Decimal Places of Natural Logarithms for Factors of the Form* $1 \pm n \cdot 10^{-p}$, *Enhanced by Auxiliary Tables of Logarithms of Small Integers*, New Haven, Conn., 1942, 120 p. [See *MTAC*, v. 1, 1943, RMT **86**, p. 20.]
7. "Natural logarithms of small prime numbers," Nat. Acad. Sci., *Proc.*, v. 29, 1943, p. 319–325. [See *MTAC*, v. 1, 1944, RMT **139**, p. 177.]
8. *Exact Values of the First 200 Factorials*, New Haven, Conn., 1944, 24 p. [See *MTAC*, v. 1, 1944, RMT **158**, p. 312.]
9. "First proof that the Mersenne number M_{157} is composite," Nat. Acad. Sci., *Proc.*, v. 30, 1944, p. 314–316.
10. "Note on the Mersenne numbers M_{167} and M_{167}," Amer. Math. Soc., *Bull.*, v. 52, 1946. p. 178.
11. "Special values of $e^{k\pi}$, $\cosh k\pi$, and $\sinh k\pi$ to 136 figures," Nat. Acad. Sci., *Proc.*, v. 33, 1947, p. 34–41.
12. "Huge numbers," Note **66**, *MTAC*, v. 2, 1947, p. 224–225.
13. "On Mersenne's number M_{227} and cognate data," Amer. Math. Soc., *Bull.*, v. 54, 1948, p. 378–380.
14. "On all of Mersenne's numbers, particularly M_{193}," Nat. Acad. Sci., *Proc.*, v. 34, 1948, p. 102–103.
15. "The magnitude of higher terms of the Lucasian sequence 4, 14, 194 ⋯," *MTAC*, v. 3, 1948, RMT **89**, p. 142–143.
16. "Twenty exact factorials between 304! and 401!," Nat. Acad. Sci., *Proc.*, v. 34, 1948, p. 407–412.
17. "A mathematician's tribute to the State of Israel," *Scripta Math.*, v. 14, 1948, p. 281–283,
18. "The Arabian Nights' factorial and the weighted-mean factorial," *Scripta Math.*, v. 15. 1949, p. 94–96.
19. "Table of exact values of high powers of 2," *Scripta Math.*, v. 15, 1949, p. 247–251.
20. "Miscellaneous hints for and experiences in computation," *Scripta Math.*, v. 16, 1950, p. 31–42.
21. "A new constant property of the parabola," *Scripta Math.*, v. 16, 1950, p. 161–167.

22. "A colossal primitive Pythagorean triangle," *Amer. Math. Mon.*, v. 57, 1950, p. 331–332.
23. "Many-figure approximations to $\sqrt{2}$, and distribution of digits in $\sqrt{2}$ and $1/\sqrt{2}$," Nat. Acad. Sci., *Proc.*, v. 37, 1951, p. 63–67.
24. "Approximations exceeding 1300 decimals for $\sqrt{3}$, $1/\sqrt{3}$, $\sin\left(\frac{\pi}{3}\right)$, and distribution of digits in them," Nat. Acad. Sci., *Proc.*, v. 37, 1951, p. 443–447.
25. "Many figure values of the logarithms of the year of destiny and other constants," *Scripta Math.*, v. 17, 1951, p. 202–208.
26. "Sequel to the note on the parabola," *Scripta Math.*, v. 18, 1952, p. 35–38.
27. "A brief history of the investigations of Mersenne numbers and the latest immense primes," *Scripta Math.*, v. 18, 1952, p. 122–131.
28. "Many-figure approximations for $3\sqrt{2}$, $3\sqrt{3}$, $3\sqrt{4}$, and $3\sqrt{9}$ with χ^2 data," *Scripta Math.*, v. 18, 1952, p. 173–176.
29. "A mathematical copying camera," *Scripta Math.*, v. 19, 1953, p. 40–44.
30. "Humorous epithets as applied to the calculation of $\sqrt{2}$," *Scripta Math.*, v. 19, 1953, p. 78–79.
31. "On the 16th and 17th perfect numbers," *Scripta Math.*, v. 19, 1953, p. 128–131.
32. "Omnibus checking of the 61-place table of denary logarithms compiled by Peters and Stein, by Callet, and by Parkhurst," Nat. Acad. Sci., *Proc.*, v. 39, 1953, p. 533–537.
33. "Exact location of the 10^mth digit in the consecutively written sequence of the natural numbers 1, 2, 3, 4, $\cdots$," *Scripta Math.*, v. 19, 1953, p. 201–204.
34. "Hamartiexéresis as applied to tables involving logarithms," Nat. Acad. Sci., *Proc.*, v. 40, 1954, p. 728–731.
35. "Full values of the first seventeen perfect numbers," *Scripta Math.*, v. 20, 1954, p. 240. (Typographical errors appear in v_8, v_{11}, and v_{13}.)
36. "Nine exact factorials between 449! and 751!," *Scripta Math.*, v. 21, 1955, p. 138–145.
37. "Exact values of 996! and 1000! with skeleton tables of antecedent constants," *Scripta Math.*, v. 21, 1955, p. 261–268.